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AN EXTENSION OF KLEIN'S ERLANGER PROGRAM: LOGIC AS INVARIANT-THEORY.*

By F. I. MAUTNER.

Introduction. It will be shown in this paper that two-valued (Boolean) mathematical logic of propositions and propositional functions (in extension) can be considered as "*invariant-theory of the symmetric group*" in the sense of Klein's Erlanger program [1] (Chapter I). Consequently an attempt is made to create the means necessary for developing it as such a "theory of notions of invariant significance and their invariant properties and relations" (Chapters II, III and IV).

The attitude taken towards logic will be dominated by a strong analogy with coordinate-geometry and its invariant-theoretic foundations. Indeed, if taken in extension, a propositional function is determined by its truth-values Boolean 0 and 1, just as a vector or tensor in geometry is determined by its coordinates. And just as the geometrical properties are shown to be independent of any particular choice of the coordinate system by proving their invariance with respect to the group of coordinate-transformations, so the logical notions and properties will be shown to be independent of any particular assignment of truth-values (the "logical coordinate system") by showing their invariance with respect to the "group of logical coordinate-transformations," which will be seen to be the symmetric group of all permutations of the domain of individual variables (assuming only one such domain).

In this connection it ought to be remembered that the group- and invariant-theoretic definition of a geometry has so far been independent of the rigorous axiomatic foundations of geometry. Accordingly two-valued logic will in the following not be based on any particular system of axioms, but taken in its "naive" form as a purely algebraic system. For the aim of this paper is not a contribution to the rigorous foundations of logic, but the application of group- and invariant-theoretic methods. And it is hoped that this will result in an invariantive calculus whose symbolism is as algebraically suggestive and easily handled as the corresponding invariantive calculus for geometry, namely tensor calculus.

* Received April 30, 1945; Revised December 15, 1945.

[1] Cf. Weyl's formulation of Klein's Erlanger program (Classical Groups, pp. 13-18) on which much of the following is based.

In order to show that two-valued logic is invariant-theory of the symmetric group we prove first that the group of automorphisms of the calculus of those propositional functions which are defined for every member of one domain C of individuals is the symmetric group $\mathfrak{S}_n$ of all permutations of the (finite or transfinite) domain C. The second fundamental characteristic of an "invariant-theory" is the existence of coordinates in which the group of automorphisms induces an isomorphic group of admissible coordinate-transformations. If one defines any particular assignment of truth-values to the propositions and propositional functions to be a "logical coordinate system," then the symmetric group $\mathfrak{S}_n$ becomes the "group of logical coordinate-transformations" with respect to which the logical notions and their properties are invariant. This can be given an exact meaning by showing that Weyl's axioms for Klein's Erlanger program (*loc. cit.*) are satisfied. Hence under the assumption of only one domain of individuals we have the result that only such properties of n^r-tuples of Boolean 0's and 1's (n = cardinal number of individuals, $r = 0, 1, 2, 3, \cdots$), are of objective logical meaning as are invariant under the symmetric group.

This result is very analogous to the well known state of affairs in geometry: Each linear geometry [2] is the invariant-theory of a group of linear transformations and could be systematically characterised as such. This suggests the same for logic: Firstly an appropriate tensor algebra (Chapter II), secondly a theory of the possible transformation laws (Chapter III) and thirdly a theory of the invariant properties of objects transforming according to these possible transformation laws (Chapter IV).

The appropriate tensor algebra is obtained by observing that there is a strong analogy between a propositional function of r arguments (and the operations of the calculus of propositional functions) and a tensor of rank r (and the operations of tensor algebra). One will therefore define a "*Boolean tensor of rank r*" to be an object which is, relative to any fixed logical coordinate system, defined by an n^r-tuple of Boolean 0's and 1's, with the transformation-law "r-fold Kronecker product" of $\mathfrak{S}_n$ with itself (Kronecker product in the sense of matrices). Then one obtains what I call "*Boolean tensor algebra*" which is isomorphic to the calculus of propositional functions (over *one* domain of individuals) as far as conjunction, disjunction, implication, negation and quantification of propositional functions are concerned. Boolean tensor algebra, though very analogous to ordinary tensor algebra, differs from it in essentially three respects: The underlying group is the

[2] We confine our analogy here to the linear geometries; the analogy between other geometries and logic seems much less deep and not so suggestive.

symmetric group, the values of the "components" of a Boolean tensor are Boolean 0 or 1 and there may be transfinitely many components. Owing to the different group there are two contractions here: Sum and product over one index are already invariant; they correspond to the two quantifiers over individual variables.

In order to find all possible transformation laws one has to study appropriate representations of the group of automorphisms. Whereas in geometry one confines oneself to matrix-representations, clearly *any* representation by a group of arbitrary transformations (a "group-realisation") is a possible transformation law here. Some properties of matrix-representations can be deduced from properties of groups with endomorphisms (via the introduction of a "representation module") in virtue of the fact that every matrix is an endomorphism of a vector-space. Similarly the fact that (as will be seen) every permutation is an automorphism of a Boolean algebra leads via the introduction of a realisation-module (= a "Boolean ring with endomorphisms") to analogous theorems on group-realisations (by means of arbitrary permutations). The analogy can be followed up further: all transitive (permutation-) realisations of any group can be obtained from its regular realisation and all transitive realisations of the (finite) symmetric group are contained in Boolean tensor space.

Next one has to define a "kind of *quantity*" to correspond to every possible transformation-law (i. e. to every possible realisation). Just as Weyl's and van der Waerden's quantities for geometry and the theory of invariants form vector spaces with a matrix-representation of the group of automorphisms g induced in them (i. e. representation-modules), so "*Boolean quantities*" form Boolean algebras with a realisation of g induced in them (i. e. realisation-modules). One can thus observe step by step *a close parallelism between the full linear group and affine or projective geometry on the one hand and the symmetric group and Boolean algebra of logic on the other.* And this close analogy reaffirms that not only can logic be considered as invariant-theory of the symmetric group, but also that group- and invariant-theoretic methods should be as fruitful here as in geometry.

Naturally one will now ask for the invariant properties of Boolean quantities. This requires the definition of an invariant property or relation and leads to the definition of *Boolean invariants* and *covariants* which are strictly analogous to the ordinary in- and covariants (in the modern sense). It is shown that the components of a Boolean covariant of any Boolean quantity q are Boolean polynomials of the components of q, no matter what the group of automorphisms. Hence there exists always a basis for the Boolean in-

variants of any Boolean quantity q such that every Boolean invariant is a Boolean polynomial of these "basic invariants," which are themselves Boolean polynomials of the components of q. This basis cannot however always be finite. Finally it is shown that the Boolean invariants (under $\mathfrak{S}_n$) of propositional functions over one finite domain of individuals can all be obtained by the processes of the calculus of propositional functions; in particular the Boolean invariants of one propositional function of one argument can be expressed by numerical conditions.

In conclusion the question as to the possibility of a further extension of Klein's Erlanger program is raised. It is indicated in what way Boolean invariants under the symmetric group might perhaps be applied to yield a classification of formal axiomatic theories. For it is characteristic of the axiomatic method to assume meaningless, hence in particular indistinguishable, individuals. Therefore, if there is only one domain of individuals, the formalism of any completely formal axiomatic theory must be invariant under all permutations of its individuals.

I should like to thank Professor Hermann Weyl for his kind assistance in obtaining very helpful criticism of the original manuscript which led to its revision.

CHAPTER I.

Invariance under the Symmetric Group.

1. The group of automorphisms of the calculus of propositional functions. We assume *one domain C of individuals* to be given as *a priori fixed*, and that *every propositional function $f(x)$ is defined for every x in C.* Every $f(x)$ is then fully determined by that subset C_1 of C such that $f(x)$ is true if and only if x is in C_1. Hence the Boolean algebra of all $f(x)$ is isomorphic to the Boolean algebra of all subsets C_1 of C (called the "full Boolean algebra of dimension n" where n is the cardinal of C).

THEOREM 1.1. *The group of automorphisms of the Boolean algebra B_n of all subsets of a set C is the symmetric group $\mathfrak{S}_n$ of all permutations of C.* (Proof obvious).

One now sees the necessity of an *a priori fixed* domain C of individual variables: Otherwise one could not speak of *the group* of automorphisms.

It follows incidentally that every automorphism of a finite Boolean algebra B_n induces and is induced by an automorphism of the lattice of all Boolean subalgebras

of B_n. For every finite Boolean algebra is full and G. Birkhoff[3] has shown that the lattice of all Boolean subalgebras is dually isomorphic to the lattice $E(C)$ of all equivalence relations over C if n is finite and that the group of automorphisms of $E(C)$ is the symmetric group $\mathfrak{S}_n$ on C.[4]

Thus the group of automorphisms of the calculus of propositional functions of one variable is—with respect to conjunction, disjunction, implication and negation—the symmetric group $\mathfrak{S}_n$. A propositional function $f(x, y)$ is determined by that subset of the totality of ordered pairs (x, y) with x and y in C, for which it takes the value 1 (true). I. e., we make the *assumption* that *the domain and converse domain of every binary relation coincide and is the same for any two* $f(x, y)$, *namely* C. Similarly *all propositional functions* $f(x_1, \cdots, x_r)$ *of* r *arguments under consideration are assumed to have the same range of definition, namely the set of all ordered* r*-tuples of elements of* C. Under this restriction the propositional functions of r arguments form with respect to conjunction, disjunction, implication and negation a Boolean algebra $[B_n]_r$ isomorphic to the Boolean algebra of all subsets of the set $[C]_r$ of all ordered r-tuples of elements of C.[5] The permutations of C induce in $[C]_r$ a subgroup of the symmetric group of all permutations of $[C]_r$, every one of which is by Theorem 1. 1 an automorphism of $[B_n]_r$. Hence $\mathfrak{S}_n$ induces in $[C]_r$ a subgroup of the group of automorphisms of $[B_n]_r$.

That quantifiers such as

$$\sum_{x_{i_1}} \prod_{x_{i_2}} \cdots \sum_{x_{i_k}} f(x_1, x_2, \cdots, x_r)$$

are invariant follows at once: Each $\sum_{x_i}$ or $\prod_{x_i}$ is a symmetric function of all the values of f as x_i ranges over C and hence it is invariant.

Again the two quantifiers

$$\sum_{f \in [B_n]_r} f(x_1, \cdots, x_r), \qquad \prod_{f \in [B_n]_r} f(x_1, \cdots, x_r)$$

ranging over all propositional functions of r arguments are clearly invariant.

Besides conjunction and disjunction of propositional functions of the same

---

[3] G. Birkhoff, *Lattice Theory* (New York 1940), Theorem 6.7. We adopt the terminology of this book.

[4] G. Birkhoff, *Proceedings of the Cambridge Philosophical Society*, vol. 31, p. 449.

[5] J. C. C. McKinsey has given an axiomatisation of the calculus of binary relations, every realisation of which is isomorphic to the Boolean algebra of the set of all ordered couples from some domain of individuals. (*Journal of Symbolic Logic*, vol. 5 (1940), p. 94, Theorem B).

variables, there are conjunctions and disjunctions of propositional functions of different variables. E. g.,

$$f(x) \,\&\, g(y) = h(x, y); \qquad f(x) \vee g(y) = k(x, y).$$

Let S be any permutation of C and denote by $S \cdot f(x)$, $S \cdot h(x, y), \cdots$ the propositional functions into which $f(x)$, $h(x, y), \cdots$ are transformed by S. I. e., let $S \cdot f(x) = f(Sx)$, $S \cdot h(x, y) = h(Sx, Sy)$ where $Sx, Sy, \cdots$ are the elements of C into which $x, y, \cdots$ are transformed by S. Then

$$S \cdot (f(x) \,\&\, g(y)) = S \cdot h(x, y) = h(Sx, Sy)$$
$$\text{and} \quad S \cdot f(x) \,\&\, S \cdot g(y) = f(Sx) \,\&\, g(Sy) = h(Sx, Sy)$$
$$\therefore \quad S \cdot (f(x) \,\&\, g(y)) = S \cdot f(x) \,\&\, S \cdot g(y).$$

Similarly

$$S \cdot (f(x) \vee g(y)) = S \cdot f(x) \vee S \cdot g(y).$$

In exactly the same way it follows that

$$S \cdot (f(x_1, \cdots, x_r) \,\&\, g(x_{r+1}, \cdots, x_t)) = S \cdot f(x_1, \cdots, x_r) \,\&\, Sg(x_{r+1}, \cdots, x_t)$$

and

$$S \cdot (f(x_1, \cdots, x_r) \vee g(x_{r+1}, \cdots, x_t)) = S \cdot f(x_1, \cdots, x_r) \vee S \cdot g(x_{r+1}, \cdots, x_t)$$

i. e. every S in $\mathfrak{S}_n$ is an automorphism also with respect to conjunction and disjunction of propositional functions of different variables.

In order to show now that such formation as the relative sum and relative product

$$\sum_{y \in C} f(x, y) \,\&\, g(y, z), \qquad \prod_{y \in C} f(x, y) \vee g(y, z)$$

of two binary relations $f(x, y)$, $g(x, y)$ are invariant, it only remains to remark that to put equal variables in a propositional function is clearly invariant under permutations of C. This completes the proof of

THEOREM 1. 2. *The group of automorphisms of the calculus of propositional functions is—under the assumption of one a priori fixed domain C of individual variables—the symmetric group $\mathfrak{S}_n$ of all permutations of C, provided every $f(x_1, x_2, \cdots, x_r)$ is uniquely defined for every x_k in C and no other x.*[6]

---

[6] The following remark is due to one of the referees: The extended group of automorphisms may be obtained by interchanging 0 and 1, i. e. by adjoining the fundamental duality of the calculus of propositional functions to its group of automorphisms, exactly as one adjoins the polarities to the group of collineations.

2. Logical coordinates; principle of relativity. In order to be able to give an exact meaning to "independence of the particular assignment of truth-values" we shall now define logical coordinates in strict analogy to the coordinates of geometry:

Suppose the system Φ of all propositional functions of r arguments x in C to be abstractly given, without any particular assignment of truth-values, for instance by formal axioms. Next introduce the system of all possible truth-functions or "logical coordinates" consisting of all n^r-tuples ξ of Boolean 0 and 1, where 0 and 1 occur together n^r times in ξ (n = cardinal of C).

Any isomorphism between the system Φ of abstractly given propositional functions f of r arguments and the system Ξ of all n^r-tuples ξ of Boolean 0 and 1 ($r = 0, 1, 2, \cdots$) is defined to be an admissible logical coordinate-system. ($r = 0$ includes atomic propositions as a special case).

There are as many isomorphisms between Φ and Ξ as there are automorphisms of Φ (or Ξ),[7] hence $n!$ different equally admissible logical coordinate-systems,[8] in the sense that if any one coordinatisation

$$f \leftrightarrow \xi$$

is given, every other equally admissible one is obtained by applying an automorphism to Φ (or Ξ). Hence the group of transitions between equally admissible logical coordinate-systems is induced by the group of automorphisms of Φ, i. e. by the symmetric group $\mathfrak{S}_n$ on C. It is the realisation of $\mathfrak{S}_n$ as the group of regular permutations of the class of all $n!$ equally admissible logical coordinate-systems (the "regular realisation of $\mathfrak{S}_n$").

An alternative way of introducing logical coordinates is the following: Again suppose the system Φ of propositional functions to be abstractly given. Let Φ_1 be the set of all propositional functions $f_k(x)$ of one argument which are true for one and only one x ($= x_k$). [To be true for a given number of individuals is of abstract objective meaning, since it is invariant]. Then any other $f(x)$ is the unique logical sum of all the $f_k(x)$:

$$f(x) = \sum_{k \in K} \mathfrak{e}_k \cdot f_k(x) \quad \text{with} \quad \mathfrak{e}_k = \begin{cases} 1 & \text{if } f(x) \supseteq f_k(x)^9 \\ 0 & \text{if } f(x) \not\supseteq f_k(x). \end{cases}$$

Similarly every $f(x_1, \cdots, x_r)$ can be expressed uniquely in the form

$$f(x_1, x_2, \cdots, x_r) = \sum_{k_1, k_2, \ldots, k_r \in K} \mathfrak{e}_{k_1 k_2 \ldots k_r} \cdot f_{k_1}(x_1) \cdot f_{k_2}(x_2) \cdots f_{k_r}(x_r)$$

---

[7] This is true for any two isomorphic algebraic systems.

[8] By $n!$ we mean the cardinal number of all permutations of n things.

[9] $f \supseteq g$ means g implies f, $f \cdot g$ means logical product of f and g.

where

$$\epsilon_{k_1 k_2 \ldots k_r} = \begin{cases} 1 \text{ if } f(x_1, \cdots, x_r) \supseteq f_{k_1}(x_1) \cdot f_{k_2}(x_2) \cdots f_{k_r}(x_r) \\ 0 \text{ otherwise} \end{cases} \quad \cdots (1)$$

determines the ϵ's uniquely and where K is an arbitrary index set with the same cardinal n as C.

This uniquely determined set of $\epsilon_{k_1 \ldots k_r} = 1$ or 0 can be taken as the logical coordinates of $f(x_1, x_2, \cdots, x_r)$. Suppose $\epsilon^{(1)}{}_{k_1 \ldots k_r}$, $\epsilon^{(2)}{}_{k_1 \ldots k_r}$ are the logical coordinates of $f^{(1)}(x_1, x_2, \cdots, x_r)$, $f^{(2)}(x_1, \cdots, x_r)$ respectively, obtained from equation (1). Clearly $\epsilon^{(1)}{}_{k_1 \ldots k_r} = \epsilon^{(2)}{}_{k_1 \ldots k_r}$ for all $k_1, \cdots, k_r$ in K if and only if $f^{(1)}(x_1, \cdots, x_r) = f^{(2)}(x_1, \cdots, x_r)$ for all $x_1, \cdots, x_r$ in C.

Hence the correspondence

$$f(x_1, \cdots, x_r) \leftrightarrow \epsilon_{k_1 \ldots k_r}$$

defined by equation (1) is one-one. Moreover it is easily seen to be an isomorphism with respect to the operations of the calculus of propositional functions.

Every one of the $n!$ permutations of C induces a unique permutation of Φ_1 and hence a transition to another one-one correspondence $f \leftrightarrow \epsilon$. Clearly the class of $n!$ coordinatisations $f \leftrightarrow \epsilon$ obtained in this way is equal to the above class of $n!$ equally admissible logical coordinate-systems $f \leftrightarrow \xi$. It follows in particular that any logical coordinate-system is determined when the coordinates of all those propositional functions $f(x)$ of one argument are given which are true for one and only one value of x. *Hence any two one-one correspondences between the set Φ_1 of all abstractly given propositional functions which are true for exactly one x and the set Ξ_1 of all n-tuples of 0 and 1 with exactly one 1 determine equally admissible logical coordinate-systems and they are the only ones.*

The specification of one such logical coordinate-system is arbitrary; at best we can determine *objectively the class of equally admissible logical coordinate-systems and ascertain the group of transitions between them.* This is the "relativity-problem" for logic. Its answer is

THEOREM 1.3 (*Principle of Relativity for Logic*). *The group of transitions between equally admissible logical coordinate-systems is induced by the symmetric group $\mathfrak{S}_n$ of all permutations of the domain C of individuals; it is the regular realisation of $\mathfrak{S}_n$.*

3. Invariant-theory of the symmetric group. The existence of a group of automorphisms and of coordinates, hence of a "principle of relativity" are

the very conditions on the ground of which one can call a geometry an "invariant-theory." Since these conditions are fulfilled here just as in geometry, it results that two-valued mathematical logic is also an invariant-theory in the sense of Klein's Erlanger program. This can, however, be given an exact meaning by *defining a mathematical theory to be an Invariant-Theory of a group g if it satisfies Weyl's axioms for Klein's Erlanger program*[10] *with g as group of automorphisms.*

We then have

THEOREM 1.4. *Two-valued (Boolean) mathematical logic of propositions and propositional functions over one domain C of individual variables is invariant-theory of the symmetric group $\mathfrak{S}_n$ of all permutations of C, provided every propositional function $f(x_1, \cdots, x_r)$ is defined for all x_k in C $(k = 1, 2, \cdots, r)$ and only such x_k.*

Proof. We have to show that Weyl's axioms are satisfied:

A. (1). (Existence of a group g). This is the symmetric group $\mathfrak{S}_n$.

A. (2). (Existence of a set of elements called coordinates and a realisation of g by means of one-one correspondences within that set). The logical coordinates as defined in **I. 2**, and the regular realisation of $\mathfrak{S}_n$ as group of logical coordinate-transformations.

B. (1). [i]: (Existence of "frames" any two of which determine a group element). Logical coordinate-systems (i. e. isomorphisms $\Phi \leftrightarrow \Xi$) are the required frames, any two of which determine an automorphism of Φ, i. e., an element of $\mathfrak{S}_n$.

[ii] (Every group element shall induce a transition from one frame to another). This has been shown in **2.**

[iii] (The group of transitions is homomorphic to g). This holds by Theorem 1. 3.

B. (2). [i] (Requirement that the objects of the theory be quantities q, i. e., relative to an arbitrarily fixed frame there is a one-one mapping of the possible values of q onto the set of coordinates). By the definition of logical coordinates, as f varies over all propositional functions of r arguments, the coordinates ξ of f vary over all possible n^r-tuples of Boolean 0 and 1 in a one-

[10] Classical Groups, *loc. cit.* or *Duke Mathematical Journal*, vol. 5 (1939), p. 491.

one manner. In particular every atomic proposition has either 0 or 1 as its coordinate.

[ii] (Under transitions to other frames the coordinates of q shall be transformed according to a realisation [11] of g). For a propositional function of r arguments this realisation is the group induced by $\mathfrak{S}_n$ in the set of all ordered r-tuples of elements of C. For an atomic proposition it is the identity-realisation of $\mathfrak{S}_n$, since the two-element Boolean algebra B_1 has no automorphism besides the identity. (In this sense atomic propositions are analogous to scalars). Thus *propositions and propositional functions* (over one domain C of individuals) *satisfy the requirements of the objects of an invariant-theory, i. e. of "quantities."*

This completes the proof.

It is an interesting consequence of the fact that the one-dimensional Boolean algebra B_1 has no automorphisms other than the identity, that to "assert a proposition," i. e., to state that an atomic proposition has the value 1, is of invariant meaning.

4. Several domains of individual variables. Suppose now that there are several domains $C_1, C_2, \cdots, C_p$ of individuals ($p =$ finite). They may be "dependent" or "independent," i. e., if x_j is in C_j and x_k in C_k, x_k may be a function of x_j or not. In two special cases one can still speak of invariance with respect to one or several symmetric groups:

1) Let $C_2, C_3, \cdots, C_p$ be *completely dependent on* C_1, by which we mean that every element x_k in C_k ($k = 2, 3, \cdots, p$) is *a priori* given as such a function of elements x_1 and/or subsets C_1 of C_1 that every permutation of C_1 induces a permutation of C_k. Then it follows at once by the same argument as before that *the group of automorphisms of the calculus of all those propositional functions which are defined for every element of a finite number of a priori fixed domains* $C_1, C_2, \cdots, C_p$ *of individuals is the symmetric group of all permutations of* C_1, *provided that* $C_2, C_3, \cdots, C_p$ *are completely dependent on* C_1.

One can now introduce logical coordinates just as in the case of one domain of individuals and show again that Weyl's axioms are satisfied.

2) If there are several *independent disjoint domains* $C_1, \cdots, C_p$, i. e.,

[11] For logic we have to take axiom B (2) in its unlimited form, where a realisation of g may be any homomorphism of g onto a permutation group and *not* merely a matrix-representation.

no *a priori* functional dependence between elements of different domains then the calculus of propositional functions with arguments in one C_k only, has by Theorem 1.2 the symmetric group $\mathfrak{S}_{n_k}$ on C_k as its group of automorphisms. Propositional functions of r variables $x_j, x_k, \cdots$ from different domains $C_j, C_k, \cdots$ will transform according to the group induced in the set of all ordered r-tuples $(x_j, x_k, \cdots)$ by $\mathfrak{S}_{n_j}, \mathfrak{S}_{n_k}, \cdots$, i. e., according to the direct product $\mathfrak{S}_{n_j} \times \mathfrak{S}_{n_k} \times \cdots$. Invariance of the operations of the calculus of propositional functions (except disjunction and conjunction of propositional functions of variables from different domains) follows in the same way as before. To establish invariance of conjunction of propositional functions of arguments in different domains, let S_1, S_2 be arbitrary permutations of C_1, C_2 respectively and let x_1 be in C_1, x_2 in C_2. Then

$$f(S_1x_1) \,\&\, g(S_2x_2) = S_1f(x_1) \,\&\, S_2g(x_2) = S_1 \times S_2(f(x_1) \,\&\, g(x_2)),$$

where $S_1 \times S_2$ is the permutation induced by S_1 and S_2 in the set of all ordered pairs (x_1, x_2). Similarly for more arguments. Hence *the operations of the calculus of all those propositional functions, which are defined for every element of several a priori fixed disjoint domains $C_1, C_2, \cdots, C_p$ of individuals, are invariant under the symmetric groups on $C_1, C_2, \cdots, C_p$ and their direct products, provided the domains are completely independent* in the above sense.

In more complicated cases of several domains no such simple result will hold.

CHAPTER II.

Boolean Tensor Algebra.

The result obtained that logic is invariant-theory of the symmetric group reaffirms that there is an analogy of fundamental importance with geometry, especially with the linear geometries, and raises the question whether one cannot create a theory of invariants as an appropriate tool for the study of logical notions and their properties, similar to the powerful tools for the study of linear geometries, namely linear algebra (especially tensor algebra), theory of group-representations and invariants. This will now be attempted: First an appropriate "tensor-algebra."

1. Definition of a Boolean tensor. There is a great similarity between a propositional function of r arguments and a tensor of rank r. The propositional functions of one argument constitute the full Boolean algebra B_n of dimension n. Since a full Boolean algebra is uniquely determined (up to

isomorphism) by its dimension (= the cardinal of C), one can (just as in the case of vector algebra) take the set of all n-tuples of Boolean 0 and 1 as a model for B_n and define the operations of Boolean algebra to be performed by performing them on the "components of these Boolean vectors." This can be done for propositional functions by associating with every individual x in C in a one-one manner a "*dimension*" of a finite- or transfinite-dimensional Boolean vector space. Then the full Boolean algebra of all propositional functions $f(x_1, \cdots, x_r)$ can be isomorphically replaced by the Boolean algebra of all "*n-dimensional Boolean tensors of rank r*" if one *defines*: *An n-dimensional Boolean tensor* $a_{j_1 \ldots j_r}$ *of rank r is a single-valued function of r arguments ranging over all elements of a given set C of cardinal n, whose values are—relative to an arbitrarily fixed logical coordinate-system—Boolean* 0 *or* 1 (" *the components*" *of the Boolean tensor*). *Under a logical coordinate-transformation* $a_{j_1 \ldots j_r}$ *is transformed into*

$$a'_{k_1 \ldots k_r} = a_{s j_1 \ldots s j_r}$$

obtained by applying the permutation S to every index j. Alternatively,[12] if (s_{kj}) is the permutation-matrix corresponding to S

$$a'_{k_1 \ldots k_r} = \sum_{j_1, j_2, \ldots, j_r} s_{k_1 j_1} s_{k_2 j_2} \cdots s_{k_r j_r} a_{j_1 j_2 \ldots j_r}$$
$$= \prod_{j_1, j_2, \ldots, j_r} (\bar{s}_{k_1 j_1} + \bar{s}_{k_2 j_2} + \cdots + \bar{s}_{k_r j_r} + a_{j_1 j_2 \ldots j_r}).$$

The last equality follows by observing that the permutation-matrix (s_{kj}) has in the row corresponding to the index i exactly one 1, at the intersection with the column i' say, and 0 elsewhere. Then $\sum_{k \in C} s_{ik} a_k = 1 \cdot a_{i'} = a_{i'}$. Hence $\prod_{k \in C} (\bar{s}_{ik} + a_k) = 0 + a_{i'}$, i. e. $\sum_{k \in C} s_{ik} a_k = \prod_{k \in C} (\bar{s}_{ik} + a_k)$ for any (finite or transfinite) permutation-matrix.

Thus the transformation law of a Boolean tensor is quite similar to that of an ordinary tensor; if a Boolean vector transforms according to the permutation S, i. e., according to the permutation matrix $[s_{jk}]$, then a Boolean tensor of rank r transforms according to the r-fold "Kronecker product" of the permutation matrix S:

$$[S]_r = [s_{j_1 k_1} s_{j_2 k_2} \cdots s_{j_r k_r}]$$

in the sense of the Kronecker product of matrices.

Since a permutation matrix is a special case of an orthogonal matrix, there

[12] We use $a + b$, ab, $\bar{a}$ for Boolean algebra-sum, product and complement respectively.

can be *no distinction between contravariant and covariant components in the case of Boolean tensors. Thus the Boolean principle of duality has—in contradistinction to geometry—its cause in a duality of the algebra of coordinates and not in a duality* (contragredience) *of the transformation group.* It also follows that there is no distinction between tensors and tensor densities here.

2. Addition, multiplication, subsumption and complementation of Boolean tensors. One has now to develop the possible invariant operations on Boolean tensors. They are quite analogous to those of ordinary tensor algebra, but owing to the fact that here the only transformations are permutations of the components, Boolean tensor algebra is richer in operations:

First one has the operations of Boolean algebra to be performed on the components of Boolean tensors of fixed rank r and yielding again Boolean tensors of rank r. E. g.,

$$a_{i_1 i_2 \dots i_r} + b_{i_1 i_2 \dots i_r} = c_{i_1 i_2 \dots i_r}.$$

Besides, two Boolean tensors $a_{i_1 \dots i_r}$, $b_{k_1 \dots k_t}$ can be added (multiplied) by adding (multiplying) *every* component of a with *every* component of b, relative to a fixed logical coordinate-system:

$$a_{i_1 \dots i_r} + b_{k_1 \dots k_t} = f_{i_1 i_2 \dots i_r k_1 k_2 \dots k_t}, \quad a_{i_1 \dots i_r} \cdot b_{k_1 \dots k_t} = g_{i_1 i_2 \dots i_r k_1 \dots k_t}$$

yielding invariantly two Boolean tensors f, g of rank $r + t$, which we shall call the *outer sum* and *outer product of the Boolean tensors a and b.* One can, of course, form the outer sum (outer product) of any number of Boolean tensors. Outer sum (or product) of an infinite number of Boolean tensors would lead to Boolean tensors of infinite rank.

Example. Let a_i, b_k be two 3-dimensional Boolean vectors; their outer sum

$$[a_i + b_k] = \begin{pmatrix} a_1 + b_1 & a_1 + b_2 & a_1 + b_3 \\ a_2 + b_1 & a_2 + b_2 & a_2 + b_3 \\ a_3 + b_1 & a_3 + b_2 & a_3 + b_3 \end{pmatrix}$$

is a Boolean tensor of rank 2.

As a particular case of an outer product one can form the outer product of r Boolean vectors:

$$a_{i_1} \cdot b_{i_2} \cdot \dots \cdot z_{i_r}$$

yielding a Boolean tensor of rank r. It follows from the transformation law of a Boolean tensor (II, 1) that *a Boolean tensor of rank r transforms like the outer product of r Boolean vectors,* just as for ordinary tensors.

However, not every Boolean tensor of rank r is of the form $a_{i_1} \cdot b_{i_2} \cdot \ldots \cdot z_{i_r}$, but it can always be expressed as sum of r-fold outer products of Boolean vectors (as is easily seen).

By duality, a Boolean tensor of rank r also transforms like the outer sum of r Boolean vectors:

$$a_{i_1} + b_{i_2} + \cdots + z_{i_r}$$

or as a combination of outer sums and products of r Boolean vectors e. g.:

$$a_{i_1} + b_{i_2} \cdot (c_{i_3} + \cdots + z_{i_r})$$

to which there is, of course, no analogy in ordinary tensor algebra.

Another invariant operation on Boolean tensors is the interchange of indices, e. g. f_{klmi} depends invariantly on f_{iklm}. In ordinary tensor space one derives invariant subspaces by imposing symmetry and anti-symmetry conditions. E. g.

$$a_{ikl} = a_{kil} \quad \text{or} \quad a_{ikl} = -a_{lki}.$$

In the case of a Boolean tensor one can still impose symmetry conditions, e. g. $f_{ijk} = f_{kji}$; however anti-symmetry conditions such as $f_{jik} = -f_{ijk}$ have no meaning here. Hence, whereas in ordinary tensor algebra there exists to every symmetry-class of tensors of rank r a complementary symmetry-class such that every (ordinary) tensor is the sum of two tensors, one from each class, this is not so for Boolean tensors.

The transformation law of a Boolean tensor $f_{i_1 i_2 \ldots i_r}$ completely symmetric in its indices (i. e., admitting every permutation of $i_1, i_2, \cdots, i_r$) is the same as that of the r-fold outer sum $f_{i_1} + f_{i_2} + \cdots + f_{i_r}$ or outer product $f_{i_1} \cdot f_{i_2} \cdot \ldots \cdot f_{i_r}$ of a Boolean vector with itself.

3. Contractions. Under the full linear group one can derive from an ordinary (mixed) tensor by "contraction" (i. e., by summation over two contragredient indices) a tensor of rank two less. Under the symmetric group one can already *sum over one single index*:

$$\sum_{j \in C} a_{i_1 \ldots i_{k-1} j i_{k+1} \ldots i_r}$$

obtaining invariantly a Boolean tensor of rank $r-1$. Dually one can form

$$\prod_{j \in C} a_{i_1 \ldots i_{k-1} j i_{k+1} \ldots i_r}$$

Thus in Boolean tensor algebra there are *two kinds of contraction*: *Sum and product over one index, each lowering the rank of the Boolean tensor by one.*

Nevertheless one can contract over several indices simultaneously. E. g.:

$$\sum_{i \epsilon C} a_{iij} = b_j \quad \text{or} \quad \prod_{i \epsilon C} c_{ijik} = d_{jk}$$

For, from a_{ikj} one obtains invariantly a Boolean tensor $f_{ij} = a_{iij}$ by putting $i = k$ in a and then contracting f_{ij}:

$$b_j = \sum_{i \epsilon C} f_{ij} = \sum_{i \epsilon C} a_{iij}$$

Besides putting indices equal one can *put indices unequal* in a Boolean tensor. E. g., let

$$b_{ik}(a_{ik}) = \begin{cases} a_{ik} & \text{for } i \neq k \\ \text{undefined} & \text{for } i = k. \end{cases}$$

Clearly, b_{ik} depends invariantly on a_{ik}. Similarly for Boolean tensors of higher rank.

Thus instead of contracting over all components of a_{ij} one can contract invariantly over all those for which $i \neq j$:

$$\sum_{i,j \epsilon C} a_{ij} \ (i \neq j) \quad \text{or} \quad \prod_{i,j \epsilon C} a_{ij} \ (i \neq j) \quad \text{or} \quad \sum_i \prod_j a_{ij} \ (i \neq j).$$

Or else over one index only. E. g.:

$$\sum_i a_{ij} \ (i \neq j), \qquad \prod_j a_{ij} \ (i \neq j).$$

4. Isomorphism with the calculus of propositional functions; Boolean tensor-equations. It is clear from the definitions of the various Boolean tensor-operations that *Boolean tensor algebra is isomorphic to the calculus of propositional functions over one* [13]*—a priori* fixed*—domain of individuals* such that to addition, multiplication, complementaton, subsumption and contractions of Boolean tensors there correspond conjunction, disjunction, negation, implication and quantification respectively of propositional functions. *Thus the calculus of propositional functions over one domain of individuals* [13] is seen to *be isomorphic to an invariantive calculus of greatest analogy to ordinary tensor algebra.*

Just as geometrical and physical laws can generally be expressed by tensor equations, so the properties of relations between the individuals of any mathematical theory which can be formalised within the calculus of propositional

[13] From I, 4 it follows that one can easily extend Boolean tensor algebra so as to be isomorphic to the calculus of propositional functions over several *fixed* domains C_k of individuals, if the transformations of each C_k are induced by the transformations of C_1.

functions over one [13] fixed domain of individuals could be expressed by invariant Boolean tensor equations.

In ordinary tensor algebra such invariant relations can be put into the form of the vanishing of a certain tensor expression, since the 0-tensor of rank r is left fixed under all linear transformations. In Boolean tensor algebra *many Boolean tensors of rank r are left fixed by* $\mathfrak{S}_n$. E. g. there are four fixed Boolean tensors of rank 2:

$$0_{ik} = 0 \text{ for all } i, k \text{ in } C;$$

$$\delta_{ik} = \begin{cases} 1 \text{ for } i = k \\ 0 \text{ for } i \neq k \end{cases} \text{(identity) and their duals } \bar{0}_{ik} = 1_{ik} \text{ and } \bar{\delta}_{ik}.$$

Similarly for Boolean tensors of higher rank. E. g.

$$f_{ijk} = \begin{cases} 0 \text{ for } i = j = k \\ 1 \text{ for } i = j \neq k \\ 0 \text{ for } i = k \neq j \\ 1 \text{ for } j = k \neq i \\ 1 \text{ for } i \neq j \neq k \neq i \end{cases}$$

is a Boolean tensor of rank 3 left fixed by $\mathfrak{S}_n$, and there are altogether 2^5 Boolean tensors of rank 3 left fixed by $\mathfrak{S}_n$, for any n. In fact it is easily seen that *all Boolean tensors of rank r left fixed by* $\mathfrak{S}_n$ *form a finite Boolean subalgebra of* $[B_n]_r$ independent of the dimension n.

Any invariant Boolean tensor expression of rank r put equal to any one of these many fixed Boolean tensors of rank r will be of invariant meaning. However the most important among them, into which any other can be brought, are

$$g_{i_1 i_2 \ldots i_r} = 0_{i_1 i_2 \ldots i_r} \text{ or equivalently } \bar{g}_{i_1 i_2 \ldots i_r} = 1_{i_1 i_2 \ldots i_r}$$

where g stands for some invariant expression in Boolean tensors and

$$0_{i_1 i_2 \ldots i_r} = 0 \text{ and } 1_{i_1 i_2 \ldots i_r} = 1 \text{ for all } i_1, i_2, \cdots, i_r \text{ in } C$$

are the "*Boolean zero-tensor*" and the "*Boolean one-tensor of rank r*" respectively. And *any mathematical proposition which can be formalised within the calculus of propositional functions over one* (*or several completely* [14] *dependent*) *domain*(*s*) *of individuals can also be expressed as such an invariant equation.*

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[14] In the sense of I, 4.

CHAPTER III.

Group-realisations.

Just as the theory of matrix-representations of (particularly linear) groups underlies the theory of invariants, tensor algebra and the development of a linear geometry as an invariant-theory, especially as it yields a knowledge of all linear transformation laws possible in the geometry in question, so one would have to study for the purpose of developing logic as invariant-theory the *realisations of a* (particularly the symmetric) *group by groups of arbitrary permutations*. But in the following we need not confine ourselves to groups:

DEFINITION. *A realisation R of an abstract semi-groupoid*[15] γ *is a homomorphism of* γ *onto a semi-groupoid* Γ *of permutations S of a set C.* The *degree* of R is the cardinal of C. The realisation R is called *faithful* if it is an isomorphism.

There is some analogy with the theory of matrix-representations: Since every matrix is an endomorphism of a vector space, one can reduce the theory of matrix-representations to the study of "representation-modules," i. e. vector spaces together with a domain of endomorphisms, i. e. groups with two domains of operators, and deduce from theorems on operator-groups theorems on matrix-representations, in particular Schur's lemma. Analogously, every permutation is by Theorem 1. 1 an automorphism of a full Boolean algebra. Hence a realisation of a semigroupoid can be considered as a full Boolean algebra (or Boolean ring[16]) together with a domain of ring-endomorphisms (a "*realisation-module*") and one can now deduce from theorems on (Boolean) rings with endomorphisms theorems on realisations, in particular an analogue of Schur's lemma (**1**).

And just as the regular (matrix-)representation of a finite group g is the source of all matrix-representations of g, so from the regular realisation (now considered as a permutation-realisation) all transitive realisations of g can be obtained.

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[15] By a semi-groupoid γ is meant an algebraic system with a product $s_1 \circ s_2 =$ element of γ (not necessarily defined for all elements), which is associative whenever defined.

[16] Stone has shown that to every Boolean algebra there is a unique Boolean ring with identity and conversely; further that every homomorphism of a Boolean algebra is one of the corresponding Boolean ring and conversely (*Transactions of the American Mathematical Society*, vol. 40 (1935).

1. Realisation-modules. We now follow Emmy Noether's example and introduce a realisation-module in analogy to her representation-module: [17]

DEFINITION. Let $R: s \to S$ be a relisation of a semi-groupoid γ by means of permutations S of a set C and let B be the Boolean algebra (or ring) of all subsets of C. For every element s of $\mathfrak{S}$ and every b of B we define a product sb by means of the equation

$$sb = Sb.$$

We call this full Boolean algebra or ring with the second multiplication sb the *realisation-module corresponding to the realisation R.*

By Theorem 1.1 the mapping $b \to sb = Sb$ is an automorphism of the Boolean algebra (and ring) B, i.e. $s(b_1 + b_2) = sb_1 + sb_2$, $s(b_1 \cdot b_2) = sb_1 \cdot sb_2$, $\overline{sb} = \overline{sb}$.

In this way one can reduce the study of realisations of γ to the study of full Boolean algebras (or rings) together with a domain Ω of ring-endomorphisms. This (commutative) "Boolean Ω-ring" determines the realisation R completely by means of the equation $sb = Sb$ and conversely. For this purpose we need the notion of an Ω-*ring* which is analogous to (but not a special case of) an Ω-group. It is a special case of the general notion of an Ω-algebra, i.e. an algebraic system relative to a given domain Ω of its endomorphisms.

DEFINITION. An Ω-*ring is a ring* $\mathfrak{R}$ *together with a domain* Ω *of ring endomorphisms* θ. A subset of $\mathfrak{R}$ is called *admissible* or an Ω-*subset* if it is transformed into itself by every θ in Ω, which defines in particular Ω-*subrings* and Ω-*ideals*. Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two Ω-rings with the same domain Ω. A mapping H of $\mathfrak{R}$ on $\mathfrak{R}'$ is called an Ω-*homomorphism* if it is a ring-homomorphism and if $H\theta = \theta H$ for every $\theta \in \Omega$.

To establish the following two theorems one has only to remark that they are true for Ω-groups and for ordinary rings.[18]

THEOREM 3.1 (*The* Ω-*homomorphism theorem for rings*). *If H is any* Ω-*ring-homomorphism of* $\Omega - \mathfrak{R}$ *on* $\Omega - \mathfrak{R}'$ *then* $\mathfrak{R}'$ *is* Ω-*isomorphic to* $\mathfrak{R}/J$, *where J is the* Ω-*ideal of all elements of* $\mathfrak{R}$ *mapped into* $0'$, *and conversely.*

COROLLARY. *The* Ω-*homomorphism theorem holds for Boolean rings and*

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[17] Cf. v. d. Waerden, "Gruppen von linearen Transformationen," II, § 1 (*Ergebnisse der Mathematik* (1935)). Our treatment is strictly analogous to the one given there.

[18] I owe this observation to one of the referees.

Boolean algebras. For Stone has shown (*loc. cit.*) that the ordinary homomorphism theorem holds for Boolean rings and that every Boolean ring-homomorphism is a Boolean algebra-homomorphism and conversely.

THEOREM 3.2 (*The Ω-isomorphism theorem for rings*). *Let J_1 and J_2 be two Ω-ideals of an Ω-ring $\mathfrak{R}$, and let $J_1 + J_2$ be their greatest common divisor, $J_1 \cap J_2$ their least common multiple. Then*

$$(J_1 + J_2)/J_1 \cong J_2/(J_1 \cap J_2)$$

where $\cong$ means Ω-isomorphic.

In analogy to the application of the theory of Ω-groups to matrix-representations one can now apply Ω-rings to (permutation-)realisations:

1) Transitivity and intransitivity: If R is an intransitive (reducible) realisation, then C has a subset C_1 invariant under R. C_1 then transforms according to a realisation R_1 (say) called a *component R_1 of R* or *contained in R*. If B_1 is the Boolean subalgebra of B of all subsets of C_1 then $\Omega - B_1$ is a principal Ω-ideal of $\Omega - B$ and conversely every principal Ω-ideal of $\Omega - B$ defines a unique component of R. Hence the (distributive) lattices of all principal Ω-ideals of $\Omega - B$ and all components of R are isomorphic. In particular to the minimal principal Ω-ideals of $\Omega - B$ corresponds the irreducible components (subsets) of R (of C). Hence R is irreducible (= transitive if γ is a group) if and only if the corresponding realisation-module is simple, i. e. has no proper principal Ω-ideals.

2) If $\Omega - B$ is the *direct sum* of principal Ω-ideals $B_1, B_2, \cdots$

$$B = B_1 \dot{+} B_2 \dot{+} \cdots \text{ and therefore } R = R_1 \dot{+} R_2 \dot{+} \cdots$$

we say that $\Omega - B$ (and R) is *decomposable*. If C is finite or γ a group then the lattice of all principal Ω-ideals of $\Omega - B$ is a full Boolean algebra; hence in particular every group-realisation is sum of its transitive components (i. e. "*decomposes completely*"), as is well known.

3) Reduction into irreducible components: To successive reduction of a realisation R into irreducible components there corresponds by Theorems 3.1 and 3.2 the formation of a series

$$B = B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_\nu = \{0\} \qquad (\nu = \text{any ordinal})$$

such that every B_λ contains its successor $B_{\lambda+1}$ as a proper maximal principal

Ω-ideal. If $\Omega - B$ decomposes completely the description can be made by a direct sum of minimal principal Ω-ideals.

4) Commutators: We first need

THEOREM 3.3. *Every homomorphism of a finite Boolean algebra (or ring) B_n into a finite Boolean algebra B'_m can be expressed as a Boolean matrix M of m rows and n columns every row of which contains at most one* 1 *(and the rest* 0*'s).*

Proof. With respect to addition a Boolean ring of finite dimension constitutes an Abelian group with every element of order 2, all of whose homomorphisms are mappings $b_i \to \overset{n}{\underset{k=1}{\Delta}} a_{ik}b_k$ where Δ stands for Boolean ring-sum and b_k are the components of an n-dimensional vector over the field $F_2 = \{0, 1\}$, a_{ik} any matrix with m rows and n columns over F_2. In order to be a homomorphism also with respect to multiplication, i. e. $\Delta a_{ik}b_kc_k = (\Delta a_{ik}b_k)(\Delta a_{ik}c_k)$ it is necessary and sufficient that a_{ik} should have at most one 1 in each row. For suppose $a_{ik_1} = a_{ik_2} = 1$ and that $a_i = 0$ when j is neither k_1 nor k_2, then $\Delta(a_{ik}b_kc_k) = b_{k_1}c_{k_1}\Delta b_{k_2}c_{k_2}$ whereas

$$(\Delta a_{ik}b_k)(\Delta a_{ik}c_k) = (b_{k_1}\Delta b_{k_2})(c_{k_1}\Delta c_{k_2}) \neq b_{k_1}c_{k_1}\Delta b_{k_2}c_{k_2}$$

for all b and c unless either $a_{ik_1} = 0$ or $a_{ik_2} = 0$. But every Boolean ring homomorphism is also a Boolean algebra homomorphism (Stone, *loc. cit.*). Moreover since there is at most one 1 in each row $\overset{k}{\Delta} a_{ik}b_k = \Sigma a_{ik}b_k$. I. e., *every homomorphism $b \to b'$ of B_n into B'_m is of the form $b \to b' = M; b$* where b, b' are the variable Boolean vectors in B_n, B'_m respectively and M is a Boolean matrix with at most one 1 in each row.

Hence if R_1 and R_2 are realisations of γ by permutations S_1, S_2 respectively and $\Omega - B_1$, $\Omega - B_2$ the corresponding realisation-modules, then any Ω-homomorphism can by Theorem 3.3 be expressed by a Boolean matrix M with at most one 1 in each row such that $M; S_1(s) = S_2(s); M$ for all s in γ.

In particular if M is a permutation-matrix then $S_2(s) = M; S_1(s); M^{-1}$ i. e. equivalent realisations correspond to Ω-isomorphic realisation-modules and conversely.

If $R_1 = R_2$ then $M; S = S; M$, i. e. the Ω-endomorphisms of the realisation-module can be expressed by those Boolean matrices with at most one 1 in each row which commute with every S, and conversely.

One can now apply theorems on Ω-rings to realisations:

i) Uniqueness: If the finite chain condition holds for the lattice of principal Ω-ideals then the (generalised) Jordan-Hölder theorem entails uniqueness of successive decomposition into irreducible components. The Krull-Schmidt theorem can be applied to entail uniqueness (up to equivalence and order) of the directly indecomposable principal Ω-ideals of $\Omega - B$ and components of R.

ii) Analogue of Schur's lemma: Combining 4) above with Theorem 3.1 gives

THEOREM 3.4 (*Analogue of Schur's lemma*). *Let R_1 and R_2 be irreducible realisations of finite degree n, m respectively of a semi-groupoid γ:*

$$R_1 : s \to S_1(s), \qquad R_2 : s \to S_2(s).$$

Any Boolean matrix M of m rows and n columns with at most one 1 in each row such that

$$M ; S_1(s) = S_2(s) ; M \quad \text{for all } s \text{ in } \gamma$$

is either the zero-matrix or a permutation-matrix. In the latter case $m = n$ and R_1 and R_2 are equivalent.

In particular *every Boolean matrix M of this type which commutes with every member S of an irreducible family of permutations of a finite set is either the Boolean 0-matrix or a permutation-matrix.*

iii) Just as for a set of matrices one can now prove without difficulty

THEOREM 3.5. *Let Γ be a* (completely reducible) *system of permutations S of a finite set C: $\Gamma = \alpha_1\Gamma_1 + \alpha_2\Gamma_2 + \cdots + \alpha_r\Gamma_r$, meaning that the irreducible component Γ_k occurs α_k times in Γ (equivalent components being identified). Then*

$$M ; S = S ; M \quad \text{for all } S \text{ in } \Gamma$$

implies for any Boolean matrix M with at most one 1 in each row that it must be of the form

$$M = \begin{pmatrix} P_1 & 0 & 0 & \cdots & 0 \\ 0 & P_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & P_r \end{pmatrix}$$

where $P_1, P_2, \cdots, P_r$ are permutation-matrices or zero-matrices of degrees $\alpha_1 d_1, \alpha_2 d_2, \cdots, \alpha_r d_r$ respectively, d_k being the degree of the irreducible component Γ_k.

As a special case one can let M in Theorem 3. 5 be a permutation-matrix and Γ a group. We then obtain as a particular case the well-known

COROLLARY. *A permutation M of a finite set C which commutes with every member of a group G of permutations of C must consist of cycles permuting only such elements of C as belong to equivalent transitive subsets of C with respect to G.* (By equivalent we mean such transitive subsets of C as transform according to equivalent transitive components of G.) Another special case is the well known fact that *the only permutations which commute with a given permutation S (both acting on the same finite set C) permute only such elements of C as belong to cycles of the same length of S.*

It should be noted that all the above argument about commutators (with at most one 1 in each row) of realisations only applies to realisations of finite degree, since every infinite Boolean algebra has non-principal ideals (Stone).

2. Kronecker products; characters. Just as the notion of the Kronecker product of matrices underlies that of an ordinary tensor, so the following notion of the Kronecker product of permutations underlies that of a Boolean tensor. Let S_1 be a permutation of a set C_1 of cardinal n_1, S_2 a permutation of a set C_2 of cardinal n_2 and $[S_{ik}{}^{(1)}]$, $[S_{ik}{}^{(2)}]$ the corresponding permutation-matrices, which can be considered to act on Boolean vectors a_i, b_j of dimensions n_1, n_2 respectively: $a_i \to \sum_k S_{ik}{}^{(1)} a_k$, $b_j \to \sum_l S_{jl}{}^{(2)} b_l$. Then the $n_1 \cdot n_2$ products $a_i b_j$ can be considered as the components of an $n_1 \cdot n_2$-dimensional Boolean vector (the outer product of a and b) which undergoes a transformation

$$a_i b_j \to \sum_{kl} S_{ik}{}^{(1)} S_{jl}{}^{(2)} a_k b_l$$

whose matrix $[S_{ik}{}^{(1)} S_{jl}{}^{(2)}]$ is the Kronecker product $[S_{ik}{}^{(1)}] \times [S_{jl}{}^{(2)}]$ of the permutation-matrices $[S_{ik}{}^{(1)}]$ and $[S_{jl}{}^{(2)}]$. Clearly $[S_{ik}{}^{(1)}] \times [S_{jl}{}^{(2)}]$ is again a permutation-matrix and the corresponding permutation $S_1 \times S_2$ is easily seen to be the permutation induced by S_1 and S_2 in the set $C_1 \times C_2$ of all ordered couples (x, y) with x in C_1, y in C_2. We call $S_1 \times S_2$ *the (Kronecker) product of the two permutations S_1 and S_2.* Then if S is the transformation of a Boolean vector, the corresponding transformation of a Boolean tensor of rank r is the r-fold product $S \times \cdots \times S = [S]_r$, say, of S with itself.

If one defines (as usual) the *character of a permutation S* of a set C to be the cardinal number of elements of C left fixed by S, then it follows at once that if $\chi_1(S)$, $\chi_2(S)$ are the characters of the two realisations R_1 and R_2

then $\chi_1(S) \cdot \chi_2(S)$ is the character of $R_1 \times R_2$. It is well known that for a finite realisation of a finite group g of order $|g|$

$$\frac{1}{|g|} \sum_{s \in g} \chi(S) = \text{number of transitive components.}$$

But, by the above, the m-fold Kronecker-product $R \times \cdots \times R = [R]_m$ of R with itself is again a finite realisation of g, if R is, and it has the character $\chi^m(S)$. Hence *the character* $\chi(S)$ *of any realisation* R *of finite degree of a finite group* g *must satisfy the denumerable sequence of conditions*

$$\frac{1}{|g|} \sum_{s \in g} \chi^m(S) = \textit{positive integer}, \qquad (m = 1, 2, 3, \cdots).$$

In the same way it follows that *if* $\chi_1(S), \chi_2(S), \cdots, \chi_\nu(S)$ *are the characters of any finite number of realisations of finite degree of a finite group* g *of order* $|g|$ *then*

$$\frac{1}{|g|} \sum_{s \in g} \chi_1^{m_1}(S) \cdot \chi_2^{m_2}(S) \cdots \chi_\nu^{m_\nu}(S) = \textit{positive integer},$$

where $m_1, m_2, \cdots, m_\nu =$ *any combination of positive integers.*

3. The regular realisation. We first recall that every transitive realisation T of a group g is equivalent to a realisation obtained by associating with every group-element s of g the permutation which the left-cosets of a certain subgroup h of g experience under left-multiplication by s. We call h the subgroup which *generates* the transitive realisation T.[19]

We next need a criterion to decide whether a given transitive realisation T of a group g is contained in another given realisation D of g:

THEOREM 3.6. *Let* $D: s \to S$ *be a faithful realisation of a group* g *by means of a group* G *of permutations* S *of a set* C *and let* T *be a transitive realisation of* g *generated by the subgroup* h *of* g. *Suppose that under the realisation* D *there corresponds to* h *the subgroup* H *of* G. *Then* T *is contained in* D *if and only if there exists at least one element* c *of* C *such that* H *is the largest subgroup of* G *which leaves* c *fixed.*

Proof. Necessity: If T is contained in D, then there exists a subset C_1 of C which is transformed into itself according to T. T is generated by a certain subgroup h of g, i. e. there is a one-one correspondence

$$c_k \leftrightarrow w_k h$$

[19] Speiser, *Theorie der Gruppen*, § 37 (2nd edition).

between the elements c_k of C_1 and the left-cosets $w_k h$ of h in g, such that $Sc_j = c_k$ if and only if $sw_j h = w_k h$. The element c corresponding to h itself under this one-one correspondence is left fixed by every permutation S in H and by no other element of G, since $sh = h$ if and only if s is in h. Hence if T is contained in D then at least one element of the subset C_1 of C is left fixed by every element of the subgroup H and by no other element of G.

Sufficiency. Conversely, if to a given subgroup h of g there exists under the one-one correspondence $D: g \rightarrow G$ (and in particular $h \rightarrow H$) an element c_1 of C such that H is the maximal subgroup of G which leaves c_1 fixed, then any two permutations in G and not in H will transform c_1 into $c_2 \neq c_1$, if and only if they belong to the same left coset of H in G. Hence there is a 1-1 correspondence between the elements c_k of C into which c_1 is transformed and the left-cosets of H in G. Choose this 1-1 correspondence so that to $c_k = W_k c_1$ there corresponds the coset $W_k H$ of H in G. Then the element S of G transforms c_k into $Sc_k = SW_k c_1$ and $W_k H$ into $SW_k H$. Hence the transformations of the elements c_k of C (into which c_1 is transformed) constitute a transitive realisation which is equivalent to that generated by the subgroup H of G, which is in turn equivalent to that generated by the subgroup h of g, since the correspondence $g \rightarrow G$ is isomorphic.

THEOREM 3.7. *Let R be the regular realisation of any group g by means of permutations of a set C. Then the realisation $\langle R \rangle$ of g induced by R in the set $\langle C \rangle$ of all subsets of C contains all transitive realisations of g.*

Proof. We have only to show that the criterion of Theorem 3.6 is satisfied. Since R is the regular realisation we can identify C with g and the permutations of C with the left-translations of g. Then we have to show that there exists to every subgroup h of g at least one subset U of g, such that U goes over into itself under left-translations by elements of h, but by no other elements of g. Clearly any right-coset hw_k of h in g is such a subset $U: shw_k = hw_k$ if and only if s is in h.

If g is the symmetric group $\mathfrak{S}_n$ of all permutations of a finite set C, then the set of all ordered $(n-1)$-tuples $(x_1, x_2, \cdots, x_{n-1})$ of elements of C, such that $x_1 \neq x_2 \neq \cdots \neq x_{n-1}$, transforms according to the regular realisation of $\mathfrak{S}_n$. But it is a principal ideal of Boolean tensor-space of rank $n-1$. Hence we have

THEOREM 3.8. *Every transitive realisation of the finite symmetric group is contained in Boolean tensor space.*

By Theorem 3.5 this is equivalent to the fact that to every subgroup H

of $\mathfrak{S}_n$ there exists, if n is finite, at least one propositional function f, such that H is the largest subgroup of $\mathfrak{S}_n$ which leaves f fixed. As a natural generalisation of an automorphism of an algebraic system, we call *the maximal group of permutations of a set C which leaves a propositional function f, defined over C, fixed, the group of automorphisms of the propositional function f*[20] and have the result:

THEOREM 3. 9. *Every group of permutations of a finite set C is the group of automorphisms of at least one propositional function defined over C.*

However, the correspondence between propositional functions and their groups of automorphisms is many—one not one-one, as is easily seen.

If one admits propositional functions of transfinitely many variables (Boolean tensors of transfinite rank) then Theorems 3. 8 and 3. 9 would hold also for a domain with transfinitely many individuals.

CHAPTER IV.

Boolean Theory of Invariants.

1. Boolean quantities. In order to develop an analogue of the theory of invariants as the appropriate instrument for the study of logical notions and their properties our first step must be the *definition of "quantities"* which can appear as arguments and values in the formation of invariant functions. I. e., we have to define for every possible transformation law its corresponding substratum. Any such quantity q is reproducibly determined relative to a fixed logical coordinate-system by an m-tuple of Boolean 0's and 1's (an m-dimensional Boolean vector), the logical coordinates q_i (say) of q. In another logical coordinate-system q will have coordinates q'_i obtained from q_i by applying the permutation S to the components q_i which corresponds to the transition s from the first logical coordinate-system to the second. The transition s is an element of the given (abstract) group of automorphisms g, and S corresponds to s under the realisation $R: s \to S$; this is the transformation-law which characterises the "*kind of the quantity q*," whereas g characterises the theory as a whole. We can without loss of generality restrict ourselves first to those q which constitute full Boolean algebras. For any collection of q's, being determined relatively to any logical coordinate-system

[20] This definition has been given in the case of binary relations by O. Ore, Galois-connections, *Transactions of the American Mathematical Society* (1944).

by Boolean vectors, will constitute a subset of a full Boolean algebra. We are thus led to the following

DEFINITION. *A Boolean quantity q of kind R is an element of a full Boolean algebra B_m of a certain dimension m which is characterised by a realisation R of the group of automorphisms $g: s \to S$ of degree m. Each value of q determines relative to a logical coordinate-system an m-dimensional Boolean vector (q_i), which transforms under the transition s to another logical coordinate-system by applying the permutation S to the components q_i.*

Hence *a system of Boolean quantities of kind R is the realisation-module* (cf. III, 1) *corresponding to the realisation R of the group of automorphisms*; just as a system of quantities in geometry is a representation-module, i. e., a vector-space with a given matrix-representation induced in it, which describes the transitions to another (geometrical) coordinate-system. Thus "Boolean quantities" are strictly analogous to Weyl's [21] and v. d. Waerden's [22] quantities for geometry.

Although we need not specify the group of automorphisms in the definition of a Boolean quantity, for the purpose of logic it will be the symmetric group.

Examples. 1) The simplest case of a Boolean quantity is an atomic proposition, in extension. The realisation-module consists of the one-dimensional Boolean algebra $\{0, 1\}$ (the truth-values) with the identity-realisation as the induced group.

2) Boolean tensor-space of rank r is a system of Boolean quantities of kind $\mathfrak{S}_n \times \cdots \times \mathfrak{S}_n = [\mathfrak{S}_n]_r$ (= the r-fold Kronecker product of $\mathfrak{S}_n$ with itself).

3) Symmetrical Boolean tensors: All n-dimensional Boolean tensors of rank r whose indices satisfy certain symmetry-conditions form a full Boolean algebra with a certain realisation of $\mathfrak{S}_n$ induced in it.

Two systems of Boolean quantities will be called *equivalent* if they correspond to equivalent realisations.

A system Q of Boolean quantities q of kind R will in general contain other systems of Boolean quantities: Any full Boolean subalgebra of Q which is transformed into itself under R will be a system of Boolean quantities contained in Q. Among them are the admissible principal ideals of Q. To each

[21] *Classical Groups* (*loc. cit.*).
[22] *Mathematische Annalen*, vol. 113 (1937), p. 15.

minimal admissible principal ideal of Q there corresponds a transitive component of R. If Q has no proper principal admissible ideal we call it a *simple system of Boolean quantities.* I. e., *a simple system of Boolean quantities corresponds to a transitive realisation of g and conversely.*

Examples. 1) $f(x)$ is a simple quantity because $\mathfrak{S}_n$ is a transitive realisation of itself.

2) A Boolean tensor of rank $r > 1$ is not a simple Boolean quantity. E. g. a Boolean tensor f_{ik} of rank 2 contains two simple Boolean quantities, namely those f which are defined for $i = k$ and those for $i \neq k$ only.[23]

Defining the *direct sum* of Boolean quantities to be the direct sum of the realisation-modules which they constitute, it follows that Boolean quantities are "directly added" simply by juxtaposition of their components (just as for quantities with respect to linear groups). Hence *every system Q of Boolean quantities is the direct sum of simple subsystems of Q,* namely of those which correspond to the minimal admissible principal ideals of Q, *with respect to any group.*

Besides addition one can define the *Kronecker product of Boolean quantities*: If q_1 and q_2 are two Boolean quantities of kind R_1, R_2 respectively, then the Boolean quantity corresponding to $R_1 \times R_2$ will be called the *Kronecker product* $q_1 \times q_2$ of the two Boolean quantities q_1 and q_2. If $q_j^{(1)}$ and $q_k^{(2)}$ are the components of q_1 and q_2 then all possible sums $\Sigma q_j^{(1)} q_k^{(2)}$ range over all the components of $q_1 \times q_2$ and so do all the products $\Pi q_j^{(1)} + q_k^{(2)}$, as is easily seen.

Example. Boolean tensor space of rank r is the r-fold Kronecker product of Boolean vector space. Since the Kronecker product of two Boolean tensors is again one it follows that the class of simple systems of Boolean quantities occurring as admissible principal ideals of Boolean tensor space is closed with respect to Kronecker multiplication (followed by decomposition into admissible principal ideals). If Boolean tensor space of rank 1 belongs to this class, then it is obviously the smallest which is closed in this sense. However, not all simple Boolean quantities with respect to $\mathfrak{S}_n$ will belong to it in general. For examples show that the transitive components of $\mathfrak{S}_n \times \cdots \times \mathfrak{S}_n = [\mathfrak{S}_n]_r$ $(r = 1, 2, \cdots)$ do not exhaust the transitive realisations of $\mathfrak{S}_n$. Even in the simplest case of $\mathfrak{S}_3$ the realisation $\mathfrak{S}_3 \to \mathfrak{S}_2$ is not contained in $\mathfrak{S}_3 \times \mathfrak{S}_3$. To obtain all transitive realisations of $\mathfrak{S}_n$, if n is finite, one has to go further:

[23] Called self-relatives and alio-relatives by Peirce (*American Journal of Mathematics*, vol. 3, p. 44).

It is not sufficient to decompose Boolean tensor space into its principal admissible ideal, but to determine its transitive subsets. (That the latter is sufficient follows from Theorem 3.8).

2. General properties of invariance. We are now confronted with the task of studying the possible invariant properties of and relations between systems of Boolean quantities. The first question must be: What is meant by an invariant property or relation?

DEFINITION OF INVARIANCE IN THE MOST GENERAL SENSE. *A property of or relation between members $x_1, x_2, \cdots, x_k$ of a finite number of sets $C_1, C_2, \cdots, C_k$ respectively, which can be expressed as a propositional function $f(x_1, x_2, \cdots, x_k)$, is invariant with respect to an abstract group g and realisations*

$$R_1 : s \to S_1(s),\ R_2 : s \to S_2(s), \cdots,\ R_k : s \to S_k(s)$$

of g as groups of permutations of $C_1, C_2, \cdots, C_k$ respectively if

$$f(x_1, x_2, \cdots, x_k) = f(S_1(s)x_1, S_2(s)x_2, \cdots, S_k(s)x_k)$$

for all s in g and every x_j in C_j $(j = 1, 2, \cdots, k)$.[24]

In the case of one set C only of cardinal n, all possible propositional functions $f(x)$ with x in C form the full Boolean algebra of dimension n. If, however, we limit our attention to those $f(x)$ which are invariant with respect to g and a realisation $R: s \to S(s)$ by permutations of C, then whenever $f(x)$ takes on a certain truth value for a certain x_0 in C it must have—by the above definition of invariance $f(x) = f(Sx)$—the same truth value for every x in C which belongs to the same set of transitivity as x_0. Also if $f(x_0) = f(y_0)$ and $g(x_0) = g(y_0)$ then $f(x_0) + g(x_0) = f(y_0) + g(y_0)$, $f(x_0) \cdot g(x_0) = f(y_0) \cdot g(y_0)$ and $\bar{f}(x_0) = \bar{g}(x_0)$ for any fixed x_0 and y_0 in C. Hence *the totality of invariant propositional functions $f(x)$ is isomorphic to the Boolean algebra of all subsets of the set whose elements are the transitive subsets of C with respect to g.* Similarly *the invariant propositional functions*

[24] In his *Grundlagen der Geometrie*, K. Reidemeister says: One could use the definition of a geometry as invariant theory of a group as a rigorous axiomatic definition if there existed a formal logical definition of invariance. Moreover such special notions as algebraic or differential in- and covariants are special cases of the above definition and so is the notion of an automorphism of an algebra.

This definition could at once be generalised to any (associative) semi-groupoid γ (instead of a group g) and realisations of γ as sets of arbitrary transformations of $C_1, C_2, \cdots, C_k$ (See the definition at the beginning of III). But no use will be made of this possibility in the following.

$f(x_1, x_2, \cdots, x_k)$ *constitute* a subalgebra of the Boolean algebra of all propositional functions of k arguments, namely *the full subalgebra consisting of those* $f(x_1, x_2, \cdots, x_k)$ *which take on the same truth value for any such ordered k-tuples* $(x_1, x_2, \cdots, x_k)$ *as are permuted amongst each other by* g. In short they constitute the subalgebra obtained by identifying any such ordered k-tuples as are permuted by g (under the realisations $R_1, R_2, \cdots, R_k$), which is possible since g is a group and therefore permutability under any realisation of g an equivalence-relation.

It follows that a knowledge of all invariant propositional functions $f(x_1, x_2, \cdots, x_k)$ is obtained by the decomposition of $R_1 \times R_2 \times \cdots \times R_k$ into its transitive components and conversely.

The invariant propositional functions $f(x_1, x_2)$ of two arguments are equivalently characterised by the condition that the Boolean matrix F (say) whose components are the values of $f(x_1, x_2)$ shall commute with the two realisations R_1 and R_2 in the sense that for every s in g we have

$$F ; S_1(s) = S_2(s) ; F$$

where $S_1(s), S_2(s)$ are the permutation matrices corresponding to s under R_1, R_2 respectively. If $R_1 = R_2$ then the propositional functions $f(x_1, x_2)$ invariant under R are those Boolean matrices which commute with every S in R.

Returning to the case of k arguments, the number of invariant propositional functions can, if $g, C_1, C_2, \cdots, C_k$ are all finite, be obtained from the characters $\chi_1(s), \chi_2(s), \cdots, \chi_k(s)$ of $R_1, R_2, \cdots, R_k$ respectivley. For the character of $R_1 \times R_2 \times \cdots \times R_k$ is $\chi_1(s) \cdot \chi_2(s) \cdot \ldots \cdot \chi_k(s)$ and therefore the number of transitive components of $R_1 \times R_2 \times \cdots \times R_k$ is

$$m = \frac{1}{|g|} \sum_{s \in g} \chi_1(s) \cdot \chi_2(s) \cdot \ldots \cdot \chi_k(s) \qquad (|g| = \text{order of } g)$$

and $2^m =$ number of invariant propositional functions $f(x_1, x_2, \cdots, x_k)$.

In particular the number of Boolean matrices which commute with two realisations R_1, R_2 (in the above sense) is

$$2 \exp \left\{ \frac{1}{|g|} \cdot \sum_{s \in g} \chi_1(s) \cdot \chi_2(s) \right\} .$$

If $R = R_1 = R_2$ is a transitive realisation then the number of commuting Boolean matrices is

$$2 \exp \left\{ \frac{1}{|g|} \cdot \Sigma \chi^2(s) = 2^l \right\}$$

where l = number of double-cosets of that subgroup in g which generates the realisation R (in the sense of III, **3**). For it is known [25] that $\frac{1}{|g|}\sum\chi^2(s)$ is the number of double-cosets of that subgroup in g which generates the transitive realisation R. Since four Boolean matrices commute with $\mathfrak{S}_n$ and since $\mathfrak{S}_{n-1}$ is the subgroup of $\mathfrak{S}_n$ which generates the realisation of $\mathfrak{S}_n$ by itself (as follows at once from Theorem 3. 6) it follows that *the number of double-cosets of* $\mathfrak{S}_{n-1}$ *in* $\mathfrak{S}_n$ *is two, i. e.*

$$\mathfrak{S}_n = \mathfrak{S}_{n-1} + \mathfrak{S}_{n-1}a\mathfrak{S}_{n-1}\,(a \in \mathfrak{S}_n, a \notin \mathfrak{S}_{n-1})$$

if $n =$ *finite* > 2. Also *if the number of double cosets of any subgroup* g' *in any finite group* g *is* 2 *and the number of cosets of* g' *in* g *greater than* 2, *then the transitive realisation of* g *generated by* g' *has no self conjugate elements.* For if the number of double cosets is 2 then there are four commuting Boolean matrices, namely 0_{ik}, $\bar{\delta}_{ik}$, δ_{ik} and 1_{ik}, none of which except the identity δ_{ik} is a permutation-matrix, since their dimension is greater than 2.

Now suppose that g' is a subgroup of any group g and R a realisation of g by a group G of permutations of a set C. Under R there corresponds to g' a subgroup G' of G. Then the partition of C into transitive subsets with respect to G is a (not necessarily proper) refinement of the partition of C with respect to G. Hence we have

THEOREM 4. 1. *Given any set* C *of elements of invariant significance with respect to a group* g, *then* C *will also be a set of elements of invariant significance with respect to any subgroup* g' *of* g *and the propositional functions* $f(x)$ [x *in* C] *invariant with respect to* g *form a full Boolean subalgebra of the full Boolean algebra of the propositional functions* $f(x)$ *invariant with respect to* g'.

This gives an exact meaning to the statement: To decrease the group increases the possible invariants.

There is another way of "deriving" one invariant-theory from another which has been used much in geometry: If g' is a subgroup of g one considers instead of the g'-invariant propositional functions $f'(y)$ the g-invariant propositional functions $f(a, y)$ of one argument more, where the set $\{a\}$ contains an element a_0 such that $s \cdot a_0 = a_0$ if and only if $s \in g'$. (E. g. $g =$ projective group, $a_0 =$ plane at infinity, $g' =$ affine group). It is very easily seen that these two ways are equivalent for any two groups $g' \leqq g$ as far as arbitrary invariant propositional functions are concerned:

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[25] Speiser, *Theorie der Gruppen*, § 39 (2nd edition).

THEOREM 4.2 (The "*adjunction argument*"). *Let $\mathfrak{A}$ and $\mathfrak{Y}$ be two realisations of a group g within sets $\{a\}$ and $\{y\}$. Suppose that g' is that maximal subgroup of g which leaves one element a_0 of $\{a\}$ fixed (under the realisation $\mathfrak{A}$). Then every g'-invariant propositional function $f'(y)$ is obtained from a g-invariant propositional function $f(a, y)$ by putting $a = a_0$, i.e. by putting $f(a_0, y) = f'(y)$.*

Proof. Consider all ordered pairs (a_0, y). Two pairs (a_0, y_1) and (a_0, y_2) will go over into each other under g if and only if there is an s in g' such that $s \cdot y_1 = y_2$.

3. Boolean invariants and covariants. Let us now apply the general definition of invariance of **2** to an invariant-theoretic study of Boolean quantities. I. e., we want to determine the possible invariant properties of, and relations between, Boolean quantities. Let $Q = \{\cdots, q, \cdots\}$ and $H = \{\cdots, h, \cdots\}$ be two systems of Boolean quantities of kind $R_1 : s \to S_1(s)$ and $R_2 : s \to S_2(s)$ respectively, with respect to a group g. A binary relation $f(q, h)$ between q and h will be invariant if $f(q, h) = f(S_1 q, S_2 h)$ for all s in g, h in H and q in Q, i. e. if $f : q \to h$ implies $S_1 q \to S_2 h$. (Using the same symbol f for the mapping $q \to h$, i. e. q is mapped into h if and only if $f(q, h) = 1$). Confining ourselves to single-valued mappings $f : q \to h$ we *define a Boolean quantity h of kind R_1 to be a "Boolean covariant" of another Boolean quantity q of kind R_2 if h is a single-valued function $h(q)$ (i. e. $f : q \to h$) such that*

$$fS_1(s) = S_2(s)f \text{ or equivalently } h(S_1 q) = S_2 h(q) \text{ for all } s \text{ in } g.$$

I. e., the mapping $q \to h$ shall commute with the two realisations R_1, R_2.

Similarly simultaneous Boolean covariants are defined.

As a special case, if H is one-dimensional and therefore R_2 the identity realisation then h ($= j$ say) is called a *Boolean invariant of q*. I. e., *a Boolean invariant $j(q)$ of q is a single-valued function $j(q)$ whose argument is a Boolean quantity q of kind R and whose value is Boolean* 0 *or* 1 *such that $j(q) = j(Sq)$ for all q and S.* Similarly a simultaneous Boolean invariant $j(q_1, q_2, \cdots)$ of several Boolean quantities is defined. Thus a Boolean invariant j is a special case of an invariant propositional function f in the general sense of **2**: The argument of j is a Boolean quantity, whereas that of f is arbitrary. *Hence the simultaneous Boolean invariants of Boolean quantities $q_1, q_2, \cdots$ are the invariant properties of and the relations between $q_1, q_2, \cdots$, while the Boolean covariants are those invariant relations which define a single-valued dependence.* I. e., the Boolean covariants $h(q)$ corre-

spond to a subclass of all simultaneous Boolean invariants $j(h, q)$, namely to those $j(h, q)$ which define a single-valued mapping $q \to h$.

Examples. 1) Let Q consist of all propositional functions $f(x)$ of one argument x. The two quantifiers $\sum_x f(x)$, $\prod_x f(x)$ are Boolean invariants of f. But so are any symmetric functions of the values of f.

2) A Boolean tensor f_{ik} of rank 2 has Boolean covariants $\sum_i f_{ik} = h_k$ and $\prod_k f_{ik} = h'_i$ for instance. But *quantifiers,* though *the simplest Boolean covariants,* are not the only ones: Clearly, any function of f_{ik}, symmetric in one index i, say, will be a Boolean vector $c_k(f)$ say, which is a Boolean covariant of f_{ik}. Any symmetric function in both indices i and k is a Boolean invariant of f. E. g. $\sum_{i_1 \dots i_4} f_{i_1 i_2} f_{i_3 i_4}$.

THEOREM 4. 3. *Let h_1 and h_2 be two Boolean quantities of the same kind $R: s \to S$ and q be a Boolean quantity of kind $R': s \to S'$. If h_1 and h_2 are Boolean covariants of q, then so are their Boolean sum, product and complement $h_1 + h_2$, $h_1 \cdot h_2$ and $\bar{h}_1$. Hence the Boolean covariants of fixed kind R of a Boolean quantity q form a Boolean algebra.*

Proof. $h_1(q)$ and $h_2(q)$ being Boolean covariants of q we have by definition

$$h_1(S'q) = Sh_1(q) \quad \text{and} \quad h_2(S'q) = Sh_2(q).$$

Then

$$h_1(S'q) + h_2(S'q) = Sh_1(q) + Sh_2(q).$$

But S is an automorphism of the system of Boolean quantities $h_1, h_2, \cdots$, therefore

$$Sh_1(q) + Sh_2(q) = S[h_1(q) + h_2(q)]$$

hence

$$S[h_1(q) + h_2(q)] = h_1(S'q) + h_2(S'q)$$

showing that $h_1(q) + h_2(q)$ is a Boolean covariant of q. Similarly it follows that

$$h_1(S'q) \cdot h_2(S'q) = [Sh_1(q)] \cdot [Sh_2(q)] = S[h_1(q) \cdot h_2(q)]$$

and

$$\bar{h}(S'q) = \overline{Sh}(q) = S\bar{h}(q)$$

i. e. that $h_1(q) \cdot h_2(q)$ and $\bar{h}(q)$ are Boolean covariants of q with h_1 and h_2.

Among the Boolean covariants of a system Q of Boolean quantities there

is a subclass of "*homomorphic Boolean covariants* $l(q)$" defined thus: If q_1 and q_2 are any two members of Q then

$$l(q_1+q_2)=l(q_1)+l(q_2),\ l(q_1\cdot q_2)=l(q_1)\cdot l(q_2) \text{ and therefore } l(\bar{q})=\bar{l}(q).$$

I. e., the mapping $q\to l$ is an Ω homomorphism of Q onto L (as defined in III, **1**). Hence by the corollary to Theorem 3. 1 *a finite-dimensional simple Boolean quantity q has only such homomorphic Boolean covariants as are equivalent to q.* It also follows at once from the considerations of III, **1** that *all homomorphic Boolean covariants of a finite-dimensional Boolean quantity q are—up to equivalence—obtained by juxtaposition of the components of such simple Boolean quantities as constitute principal admissible ideals of Q.*

Example. f_{ii} is a homomorphic Boolean covariant of f_{ik} and so is f_{ik} $(i\neq k)$.

Since every minimal principal admissible ideal of Boolean tensor space consists of those Boolean tensors whose indices satisfy a maximum number of consistent equality and inequality conditions (as is easily seen), it follows that *all homomorphic Boolean covariants of a finite-dimensional Boolean tensor $f_{i_1 i_2 \ldots i_r}$ are—up to equivalence—obtained by imposing equality and inequality conditions on its indices.* In particular, *a finite-dimensional Boolean vector has only such homomorphic Boolean covariants as are equivalent to it.*

The *central problem* is now; *To obtain all Boolean in—and covariants of a given system of Boolean quantities.*

For Boolean invariants the general answer is almost trivial: *The Boolean invariants $j(q)$ of a system Q of Boolean quantities q constitute a full Boolean algebra isomorphic to the Boolean algebra of all subsets of the totality of transitive subsets of Q.* This follows at once from IV, **2**. In particular if g is of finite order $|g|$ and $\chi(s)$ the character of the realisation $\langle R\rangle$ induced in the set of all subsets of that set on which R acts ($R=$ kind of q), then the number of transitive subsets of Q is $\frac{1}{|g|}\sum_s \chi(s)$ and therefore the number of Boolean invariants $j(q)$ is $2\exp\left\{\frac{1}{|g|}\sum_s \chi(s)\right\}$.

In every given case one will, however, want to go a step further: To determine, if possible, a set of "*basic Boolean invariants*" explicitly as Boolean polynomials of the components of q, such that all Boolean invariants of q are generated by them. To show that there are always "Boolean basic invariants," we need

THEOREM 4. 4. *Suppose that an n-dimensional full Boolean algebra B_n*

($n =$ arbitrary cardinal) is given as the n-dimensional Boolean vector space. Then any single-valued mapping μ of any subset U of B_n onto the one-dimensional Boolean algebra B_1

$$\mu: x \to \mu x = 0 \text{ or } 1 \qquad (x \text{ in } U)$$

can be expressed as a Boolean polynomial $P(x_k)$ of the components x_k of the Boolean vector x. By a Boolean polynomial of the x_k is meant any expression in the x_k obtained by Boolean addition, multiplication and negation (including transfinite sums and products).

Proof. The mapping μ determines a partition of the subset U of B_n: $U = U_1 + U_0$ such that

$$\mu x = \begin{cases} 1 \text{ if } x \in U_1 \\ 0 \text{ if } x \in U_0. \end{cases}$$

Every Boolean vector a in B_n determines a partition $K = K_1(a) + K_0(a)$ of the arbitrary index-set $K = \{k\}$ of cardinal n such that

$$a_k = \begin{cases} 1 \text{ for } k \text{ in } K_1(a) \\ 0 \text{ for } k \text{ in } K_0(a). \end{cases}$$

Now for every a in U_1 we construct the Boolean polynomial

$$P_a(x_k) = \prod_{k \in K_1(a)} x_k \cdot \prod_{k \in K_0(a)} \bar{x}_k$$

of the components x_k of x. Clearly

$$P_a(x_k) = \begin{cases} 1 \text{ if } x = a \\ 0 \text{ if } x \neq a. \end{cases}$$

We now take the sum

$$\sum_{a \in U_1} P_a(x_k) = P(x_k), \text{ say}$$

obtaining a Boolean polynomial $P(x_k)$ of the components x_k of the variable Boolean vector x in B_n with value 1 if x is in U_1 and 0 otherwise. I. e.

$$P(x_k) = \begin{cases} 1 \text{ if } \mu x = 1 \\ 0 \text{ if } \mu x = 0 \text{ or undefined.} \end{cases}$$

COROLLARY. *Every Boolean invariant $j(q)$ of an arbitrary Boolean quantity q with respect to any group g is a Boolean polynomial of the components of q.* Hence in particular $j(q)$ *is always a sum of certain Boolean polynomials of the components of q,* namely of those which have the value 1 for every element of one transitive subset of Q and 0 elsewhere.

We shall call these Boolean invariants of q of which all others are sums the *strict basic Boolean invariants of* q.[26] Their number is equal to the number of transitive subsets of Q, hence cannot always be finite. There may be smaller bases than this strict basis, such that every $j(q)$ is a Boolean polynomial of these basis invariants, but no longer a sum of them. However, even a smaller basis for the Boolean invariants of q will be finite if and only if the number of transitive subsets of Q is finite, because it is known that the number of Boolean polynomials generated by a finite number of symbols is always finite.[27] Hence *if the number of transitive subsets of* Q *is transfinite there can be no finite basis for the Boolean invariants of* q.

Any single-valued mapping $B_n \rightarrow B_m$ can, if both B_n and B_m are taken as Boolean vector spaces, be considered as m single-valued mappings $B_n \rightarrow B_1$ of the n-dimensional Boolean algebra B_n onto the one-dimensional, namely as the mappings of B_n onto each of the m components of the variable Boolean vector of B_m (m and n arbitrary cardinals). By Theorem 4. 4 each of these m mappings can be expressed by a Boolean polynomial. Hence

THEOREM 4. 5. *Any single-valued mapping* $B_n \rightarrow B_m$ *of an* n*-dimensional Boolean vector space into an* m*-dimensional Boolean vector space* (m, n *arbitrary cardinals*) *can be expressed by letting each component of the variable Boolean vector of* B_m *be a Boolean polynomial of the components of the variable Boolean vector of* B_n. *Hence, in particular, the components of any Boolean covariant of a Boolean quantity* q *are Boolean polynomials of the components of* q.

As an immediate consequence of Theorem 4. 2 one has

THEOREM 4. 6. *The adjunction argument holds for the Boolean invariants of any Boolean quantity* q *with respect to any two groups* $g' \subseteq g$.

Indeed one has only to repeat the remark made in IV, **3** that the Boolean invariants of q are the invariant propositional functions of q.

4. The Boolean invariants of propositional functions. A Boolean polynomial in the components x_k of a Boolean vector will be called *homogeneous of degree* d if it is the sum of monomials

$$\prod_{k \in K_d} x_k \qquad (K_d = \text{arbitrary index set of cardinal } d)$$

[26] By the principle of duality it follows at once that every $j(q)$ (except $j(q) = 1$ all q) can also be expressed as the product of certain Boolean invariants of q.

[27] *Lattice Theory*, Theorem 6. 8 (p. 93).

each of degree d. It follows at once that every strict basic Boolean invariant of any Boolean quantity q of kind R is the product of a homogeneous Boolean invariant and the complement of a homogeneous Boolean invariant of q.

From now on we suppose that the *dimension of q is finite.* Then every homogeneous Boolean invariant of degree $\leq d =$ finite of q is of the form

$$\text{(i)} \qquad \sum_{k_1 \ldots k_d} P_{k_1 \ldots k_d} \cdot q_{k_1} \cdot \ldots \cdot q_{k_d}$$

where $P_{k_1 \ldots k_d}$ must be the components of a fixed member of the system of Boolean quantities of kind $R \times R \times \cdots \times R = [R]_d$. But the fixed members of $Q \times Q \times \cdots \times Q = [Q]_d$ form a full Boolean algebra each of whose atoms [28] is the Boolean sum of all members of one transitive subset of $[Q]_d$. Therefore every homogeneous Boolean invariant of degree $\leq d$ is the sum of certain expressions of the form (i) for each of which $P_{k_1 \ldots k_d}$ has a minimal number of values equal to 1. Since by the above every Boolean invariant of q is a Boolean polynomial of homogeneous one's, *these minimal homogeneous Boolean invariants form a basis for all Boolean invariants of a finite-dimensional Boolean quantity q.*

If q is a Boolean tensor $a_{ikl\ldots}$ of rank r then every homogeneous Boolean invariant (under $\mathfrak{S}_n$) of degree d of a is of the form

$$\sum P_{i_1 k_1 l_1 \ldots i_d k_d l_d \ldots} \cdot a_{i_1 k_1 l_1 \ldots} \cdot \ldots \cdot a_{i_d k_d l_d \ldots}$$

where $P_{i_1 k_1 l_1 \ldots i_d k_d l_d \ldots}$ must be the components of a *fixed* Boolean tensor of rank rd. But for such P to form the product $P_{i_1 k_1 l_1 \ldots i_d k_d l_d \ldots} \cdot \ldots \cdot a_{i_1 k_1 l_1 \ldots}$ is the same as to impose equality and inequality conditions on the indices of $a_{i_1 k_1 l_1 \ldots} \cdot a_{i_2 k_2 l_2 \ldots} \cdot \ldots \cdot a_{i_d k_d l_d \ldots}$; similarly for several Boolean tensors. *Hence every homogeneous Boolean invariant of finite-dimensional Boolean tensors $a_{ikl} \cdots$, $b_{ikl} \cdots$, $\cdots$ under $\mathfrak{S}_n$ is obtained by forming outer products, imposing equality and inequality conditions on the indices and summing over all indices* (*i. e. contracting*). By negation and multiplication all strict basic invariants are obtained from homogeneous ones, whence all invariants by addition. *Thus all Boolean Invariants under $\mathfrak{S}_n$ of Boolean Tensors of Finite Dimension* (*Propositional Functions over one finite Domain of Individuals*) *can be obtained by the processes of Boolean Tensor Algebra* (*Calculus of Propositional Functions*).

Incidentally, if one is interested in the Boolean invariants of a Boolean

[28] Meaning they cover 0.

tensor $a_{ikl}\cdots$ under a subgroup g of $\mathfrak{S}_n$ one has by the adjunction argument (Theorems 4. 2 and 4. 6) only to find a Boolean tensor f_0 which is left fixed by g and no larger subgroup of $\mathfrak{S}_n$ and to determine the simultaneous Boolean invariants $j(a, f)$ under $\mathfrak{S}_n$ and finally put $f = f_0$. That f_0 always exists follows from Theorem 3. 8 or 3. 9.

Applying the above to Boolean vectors yields at once the result that *the polarised elementary symmetric functions*

$$\sum a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot b_{k_1}\cdot b_{k_2}\cdot\ldots\cdot c_{l_1}\cdot c_{l_2}\cdot\ldots \quad \text{(any two indices} \neq)$$

form a basis for the simultaneous Boolean invariants of finite-dimensional Boolean vectors $a_k, b_k, \cdots, c_k$ *under* $\mathfrak{S}_n$. In particular a basis for the Boolean invariants of one finite-dimensional Boolean vector consists of the *elementary symmetric functions*

$$\sigma_m = \sum_{k_1<k_2<\cdots<k_m} a_{k_1}a_{k_2}\cdots a_{k_m} \quad (m = 1, \cdots, n)$$

where $\sum$ stands (as everywhere) for Boolean algebra-sum.

It is now easily seen that the Boolean algebra generated by $\sigma_1, \sigma_2, \cdots, \sigma_n$ has the following $n+1$ atoms:

$$\begin{aligned}
\psi_0 &= \bar{a}_1\bar{a}_2\cdots\bar{a}_n\\
\psi_1 &= a_1\bar{a}_2\cdots\bar{a}_n + \bar{a}_1a_2\bar{a}_3\cdots\bar{a}_n - \cdots + \bar{a}_1\bar{a}_2\cdots\bar{a}_{n-1}a_n\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\psi_m &= a_1a_2\cdots a_m\bar{a}_{m-1}\cdots\bar{a}_n + \cdots + \bar{a}_1\bar{a}_2\cdots\bar{a}_{n-m}\,\bar{a}_{n-m-1}\cdots a_n\\
&\cdots\cdots\cdots\cdots\cdots\\
\psi_n &= a_1a_2\cdots a_n.
\end{aligned}$$

Hence these $n+1$ *Boolean polynomials constitute a strict basis for all Boolean invariants of a finite-dimensional Boolean vector* $(a_1, a_2, \cdots, a_n)$ *under the symmetric group* $\mathfrak{S}_n$ *in the sense that every non-zero Boolean invariant of a is a sum of some of them.* This follows also directly by simply remarking that ψ_m has the value Boolean 1 if and only if a_k has exactly m components equal to 1.

Interpreting a_k as truth-values and therefore a as a propositional function $f(x)$ of one argument x with finite range, then ψ_m can be interpreted as the proposition: "$f(x)$ is true for exactly m values of x." Similarly σ_m becomes the proposition: "$f(x)$ is true for at least m values of x." These are the so-called numerical conditions.[29] The two sets of basic Boolean in-

[29] Hilbert-Bernays, *Grundlagen der Mathematik*, vol. I, p. 169 ff. (called *Anzahlbedingungen* there).

variants of a_k can therefore be interpreted thus: *Any proposition about one propositional function $f(x)$ of one argument x over a finite domain of individuals, which is at all of invariant meaning, can be expressed by numerical conditions.*

Another basis for the Boolean invariants of one finite-dimensional Boolean vector is obtained by applying the fundamental theorem on symmetric functions (which is true for any commutative ring with identity) to Boolean rings: *The elementary symmetric functions*

$$\overset{\Delta}{\underset{k_1<k_2<\cdots<k_m}{}} a_{k_1} \cdot a_{k_2} \cdot \ldots \cdot a_{k_m}$$

formed by means of Boolean ring-sum $a \Delta b$ (and product) constitute a basis for the Bolean $\mathfrak{S}_n$-invariants of one finite-dimensional Boolean vector.

5. On the possibility of an invariant-theoretic classification of mathematical theories. The result that logic is invariant-theory of the symmetric group places logic in a similar position for all mathematical theories (which are based on two-valued logic) to that of projective geometry for the linear geometries: The projective group is the largest linear group and therefore every other linear group a subgroup of it; hence any linear geometry can be "derived" from projective geometry by diminishing the group or, what amounts to the same, by demanding the additional invariance of one or several tensors. Similarly the symmetric group is the largest transformation group, the group of automorphisms of any mathematical theory its subgroup. This raises the question: Can one obtain mathematical theories from logic as geometries are obtained from projective geometry? I. e., can one identify mathematical theories as invariant-theories of subgroups of the symmetric group or as invariant-theories of the symmetric group relative to a "distinguished" Boolean tensor?

However, to be invariant-theory of a given group cannot lead to the definition of any mathematical theory unless an algebra of coordinates is given as well. For from IV, **2** it follows that to demand invariance means merely to consider *all* propositional functions with the same truth-values for arguments permuted into each other, hence mere invariance cannot give rise to any process of decision whether a propositional function is true or false for given individuals. Indeed if f is an invariant propositional function, then so is $\bar{f}$. Invariance is only an upper limit.

The question as to a further extension of Klein's Erlanger program must therefore be put thus: Can one identify any mathematical theory in the usual sense as invariant-theory of a subgroup of the symmetric group with two-

element Boolean algebra as algebra of coordinates? The answer seems to be in the negative. A subgroup of the symmetric group is in general the group of automorphisms of many propositional functions $f(x, \cdots)$, $g(x, \cdots, \cdots$ [cf. III, **3**], i. e., the group of automorphisms of any mathematical theory in which $f(x, \cdots)$ and/or $g(x, \cdots)$ and/or $\cdots$, are primitive ideas. Hence with B_1 as algebra of coordinates invariance with respect to a subgroup of the symmetric group would in general be ambiguous. Moreover the group of automorphisms of many mathematical theories is so small that it cannot characterise the theories in question. [E. g. The theory of numbers has only the identity as automorphism and so has the theory of real numbers. The complex numbers have only one automorphism besides the identity.] The way towards a group- and invariant-theoretic classification of mathematical theories seems to lie in a different direction:

Instead of considering subgroups of $\mathfrak{S}_n$, we keep $\mathfrak{S}_n$ as the underlying group. Indeed a mathematical theory in its completely formal abstract shape is the theory of individuals from whose nature one has entirely abstracted. I. e., (if we assume only one domain of individuals) any two individuals are a priori equivalent, indistingushable, i. e. any permutation of the individuals must be admissible. Hence *any* mathematical theory when completely formalised, must be invariant under the symmetric group $\mathfrak{S}_n$ of all permutations of its domain of individuals, although its group of automorphisms need not be $\mathfrak{S}_n$.

Example. Theory of numbers: The only automorphism is easily seen to be the identity. However if one puts Peano's axioms in the form where no special individual symbols with a distinguished meaning (e. g. zero) or special symbols for distinguished relation (e. g. successor or less than) occur, then it follows at once that each axiom is invariant under permutations of the individuals. Such a system of axioms has been given by Hilbert-Bernays.[30] One of these axioms is

$$Ad(a, b, c) \,\&\, Sq(b, r) \,\&\, Sq(c, s) \rightarrow Ad(a, r, s)$$

where Ad and Sq are arbitrary propositional functions of three and two variables respectively (interpretable as addition and successor) and a, b, c, r, s arbitrary individual variables. Let $f_{\kappa\lambda\mu}, g_{\nu\rho}$ be two arbitrary Boolean tensors of rank 3 and 2 respectively. Then the above axiom can be put in the form

$$\overline{f_{\kappa\lambda\mu} \cdot g_{\lambda\nu} \cdot g_{\mu\rho}} + f_{\kappa\nu\rho} = 1_{\kappa\lambda\mu\nu\rho} \quad (= \text{the "Boolean one-tensor" of rank 5})$$

[30] Axiomen-system Z^{**} (*Grundlagen der Mathematik*, vol. I, pp. 465-467).

or

$$\prod_{\kappa,\lambda,\mu,\nu,\rho} (\bar{f}_{\kappa\lambda\mu} + \bar{g}_{\lambda\nu} + \bar{g}_{\mu\rho} + f_{\kappa\nu\rho}) = 1.$$

In this form the left-hand side is at once seen to be a simultaneous Boolean invariant of f and g with respect to $\mathfrak{S}_n$.

As another example take the axiom for mathematical induction in the form where the special symbol for zero has been eliminated, from the same system of axioms:

$$(z)\overline{Sq}(z, b) \,\&\, A(b) \,\&\, (x)(y)[Sq(x, y) \,\&\, A(x) \rightarrow A(y)] \rightarrow A(a)$$

where A is a variable propositional function of one argument. This can be put into the form

$$\sum_{\kappa,\lambda} (f_{\kappa\lambda} + \bar{A}_\lambda) + \sum_{\mu,\nu} f_{\mu\nu} \cdot A_\mu \cdot \bar{A}_\nu + \prod_{\rho} A_\rho = 1$$

in which the left-hand side is a simultaneous Boolean invariant with respect $\mathfrak{S}_n$ of the Boolean tensor f and the variable Boolean vector A.

In general the axioms of any completely formal axiomatic theory will—by the very nature of such a theory—have to be invariant under any permutation of the individuals, if there is only *one* domain of individual variables. By applying the universal quantifier to any free individual variables, the axioms become atomic propositions i. e., Boolean invariants under $\mathfrak{S}_n$ of the Boolean tensors which corerspond to the primitive ideas of the theory and (possibly) of one or several variable Boolean tensors. *Thus a knowledge of all simultaneous Boolean invariants under $\mathfrak{S}_n$ of propositional functions $f, g, \cdots$ would yield a systematic knowledge of all possible completely formalised axiomatic mathematical theories in whose axioms $f, g, \cdots$ occur as primitive ideas or variable propositional functions.* E. g. the determination of all Boolean invariants under $\mathfrak{S}_n$ of one propositional function of three variables would yield a knowledge of all those theories of which the theory of groups, the theory of quasi-groups etc. are special cases. The knowledge of all simultaneous Boolean invariants of two propositional functions, each of three variables, would give an enumeration of all those theories of which the theory of rings, the theory of integral domains or of fields etc. are special cases.

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THE ADIABATIC LINEAR OSCILLATOR.*

By AUREL WINTNER.

1. In the linear differential equation $x'' + fx = 0$, let the coefficient $f = f(t)$ be defined and continuous on an half-line, $t_0 < t < \infty$. The coefficient function and the integration constants will be restricted to the real field. Unless stated or implied otherwise, only the non-trivial solutions $x(t)$ (that is, those which do not vanish identically) will be considered.

In a posthumous note edited by Goursat, Fatou [1] has arrived, from two different points of departure, at the following conclusion: If $l > 0$ and $L < \infty$, where

$$l = \liminf_{t\to\infty} f(t) \quad \text{and} \quad L = \limsup_{t\to\infty} f(t), \tag{1}$$

then

$$x(t) = O(1) \quad \text{as} \quad t \to \infty \tag{2}$$

holds for every solution of

$$x'' + f(t)x = 0. \tag{3}$$

The first of Fatou's starting points suggesting this theorem is nothing but an appeal to Sturm's comparison theorem; $x'' + (L + \epsilon)x = 0$, where $L + \epsilon = \text{Const.} > L$, being of higher, and $x'' + (l - \epsilon)x = 0$, where $l - \epsilon = \text{const.} > 0$, of lower, frequency than (3), as $t \to \infty$. Somewhat more details are sketched with regard to the second approach. The latter consists in a formal introduction of "polar coordinates," $x = r\cos\theta$, and of splitting (3) into a non-linear differential equation of the second order for $r = r(t)$ and into a subsequent quadrature determing $\theta = \theta(t)$. This formalism is the same as that in Newton's method for determining the radius vector $r = r(t)$ of a path described under a central (conservative) force in an (x, y)-plane. Correspondingly, there is little doubt that this second approach suggested itself to Fatou in connection with his last long paper [2], dealing with astronomical models which depend on a non-central (conservative) force.

Both of Fatou's starting points, the "Sturmian" and the "Newtonian," will be utilized in the present paper, the problem of which is due to the fact that the assertion of Fatou's posthumous note is false (that is, (2) need *not* hold if the limits (1) are positive and finite, respectively). In order to see

* Received December 20, 1945.

this, it is sufficient to choose $f(t) = p + q\cos t$, where p and q are constants satisfying $0 < q < p$, and then to apply Poincaré's results concerning the stable or unstable nature of the characteristic exponents of the resulting Mathieu equation,

$$x'' + (p + q\cos t)x = 0; \tag{3 bis}$$

cf. [7]. Quite a different counter-example has been given by Perron [4], who, without emphasizing it, has disproved *more* than what has been claimed in Fatou's note. In fact, whereas $0 < q < p$ and (1) imply that $0 < l < L$ in the case (3 bis) of (3), a glance at Perron's example shows that the two positive limits (1) coincide in his case, although (2) is still false.

In order to avoid an interruption of subsequent calculations, let here be derived a slight simplification of Perron's counter-example, in a way which will admit extensions to more sophisticated situations. Curiously enough, the natural way of deriving counter-examples of any *possible* type (cf. below) proves to be precisely Fatou's "Newtonian" approach, $x = r\cos\theta$, with $\theta = t$.

If r is independent of t, then

$$x = r\cos t \tag{4}$$

is a solution of the pure harmonic equation $x'' + x = 0$. In order to violate (2), let $x'' + x = 0$ be replaced by

$$x'' + (1 + \phi)x = 0, \tag{5}$$

where $\phi = \phi(t)$ is a continuous function satisfying

$$\phi(t) \to 0 \text{ as } t \to \infty. \tag{6}$$

Then $f(t) = 1 + \phi(t)$ in (3) is a continuous function for which the two limits (1) coincide. But for any function $r = r(t)$ (of class C'', or having just a second derivative),

$$x'' + x = r''\cos t - 2r'\sin t$$

is an identity by virtue of (4) alone. Hence, (4) will be a solution of (5) if $\phi(t)$ is defined by

$$-\phi = (r'' - 2r'\tan t)/r. \tag{7}$$

And what is now required is that $r = r(t)$ be so chosen as to make (7) a function which remains continuous (even when $\tan t$ becomes infinite) and satisfies (6).

The simplest choice of an $r = r(t)$ which fulfills these conditions but for which the function (4) violates (2) is the choice

$$r = t + \tfrac{1}{2}\sin 2t, \tag{8}$$

(which differs from Perron's example only in that it curtails the formal differentiations). In fact, (8) shows that $r' = 2\cos^2 t$. Hence, both $r' \tan t$ and r'' are continuous and bounded. But (8) also shows that $r \to \infty$ as $t \to \infty$. It follows therefore from (7) that $\phi(t)$ is continuous and satisfies (6). Nevertheless, (4) does not satisfy (3), since $r(t) \to \infty$.

2. There are various senses in which the linear oscillator (3), with a (positive) frequency f^2 which is a given continuous function of time, can be thought of as representing an *adiabatic* distortion of the conservative linear oscillator $x'' + \omega^2 x = 0$, with a constant frequency $\omega > 0$. From the point of view of Hilbert's space, the definition of an adiabatic behavior is represented by the condition

$$\int^{\infty} (f(t) - \omega^2)^2 dt < \infty$$

(to be satisfied by *some* $\omega = \text{const.} > 0$). But all that is available in this regard is contained in Weyl's linear perturbation theory in the (L^2)-space (cf. [5]). And his spectral theory of (3) deals, in the main, with *solutions* $x(t)$ satisfying the (L^2)-condition

$$\int^{\infty} x^2(t)\, dt < \infty,$$

whether the preceding (L^2)-condition, that concerning the *coefficient* $f(t)$ of (3), be satisfied or not; cf. [6].

Two other definitions of "$f(t)$ is close to $\omega^2 = \text{const.}$", namely,

$$f(t) \to \omega^2 \text{ as } t \to \infty$$

and

$$\int^{\infty} |f(t) - \omega^2|\, dt < \infty,$$

were constrasted in [8]. Clearly, none of the three conditions implies either of the other two, the three conditions being based on the metrics of the respective function spaces (L^2), (L^∞), (L).

In [8], the (L^∞)-problem was just mentioned, in connection with the Poincaré-Perron theory of (3) in the case

$$\lim_{t\to\infty} f(t) = c \neq 0, \tag{9}$$

(where, however, *c must not* be of the "stable" form ω^2; cf. [3], pp. 158-160), and was formally contrasted with an (L)-theory. If the unit of time is so chosen that ω becomes 1, the result of this (L)-theory can be formulated as follows:

Suppose that $\phi(t)$ is a continuous function satisfying

$$\int^{\infty} |\phi(t)|\, dt < \infty. \tag{10}$$

Then, corresponding to every solution, $x(t)$, of (5), *there exists a solution, $a\cos(t-\alpha)$, of the trivial approximation, $x''+x=0$, to* (5) *in such a way that*

$$x(t) - a\cos(t-\alpha) \to 0 \text{ as } t\to\infty. \tag{11}$$

In addition, the formal differentiation of this asymptotic relation is legitimate, that is,

$$x'(t) + a\sin(t-\alpha) \text{ as } t\to\infty. \tag{12}$$

Conversely, if any solution of the trivial approximation, that is, any pair of constants a, α, is given, then (5) *has a unique solution $x(t)$ satisfying* (11) *and* (12).

Actually, only the first two of these three assertions were proved in [8]. However, the third assertion, that concerning the converse, is a corollary of the first two. In fact, let $x=x_1(t)$ and $x=x_2(t)$ be two solutions of (5). By the first two assertions, there exist pairs of constants, say a_1, α_1 and a_2, α_2, satisfying (11) and (12) for $x=x_1$ and $x=x_2$ respectively. But, since (5) is self-adjoint, the (binary) Wronskian of x_1 and x_2 is independent of t. If C denotes its value, the constant C is 0 only if x_1 and x_2 are linearly dependent. On the other hand, if the asymptotic formulae (11), (12) are substituted into the Wronskian of x_1 and x_2, it is seen that $C=0$ if and only if $a_1a_2=0$. Hence, the two solutions are linearly independent if and only of $a_1\neq 0$ *and* $a_2\neq 0$. Accordingly, the converse assertion of the theorem is clear from the principle of superposition.

It is worth nothing that the theorem dualizes the spaces (L^1) and (L^∞), the assumption (10) being in the former, whereas the assertions (11), (12) concern the latter. But this is quite an accident. In fact, easy examples show that the situation fails for conjugate spaces (L^p), (L^q) if $q=p/(p-1)$

> 1 (in fact, even if $p = q$). In addition, the theorem becomes false if (L^∞) and (L) are interchanged.

The purpose of the following considerations is to collect a few general facts, positive and negative, which result if this interchange of (L^∞) and (L) is effected; that is, if the (L^∞)-condition is the *assumption* for the perturbation, $\phi(t)$, of *the coefficient function of* (5), rather than the *assertion* for the perturbations, (11) and (12), of *the solutions of* (5). In this sense, what will be developed is the (L^∞)-theory, whereas the theorem italicized above contains the (L)-theory, of the adiabatic variations of a linear oscillator.

3. Let a function $x(t)$, defined and having a continuous second derivative on an half-line $t_0 < t < \infty$, be called *quasi-harmonic* if its graph imitates the behavior of the real non-trivial solutions,

$$a \cos \omega t + b \sin \omega t, \qquad (a^2 + b^2 > 0),$$

of $x'' + \omega^2 x = 0$, where $\omega^2 = \text{const.}, > 0$, in the following sense: There exists a sequence $t_0 < u_1 < u_2 < \cdots$ such that the distance $u_{n+1} - u_n$ tends, as $n \to \infty$, to a finite, positive limit $(= \pi/\omega)$, and $x = x(t)$ is positive and concave or negative and convex (from below) according as t is on the open intervals (u_1, u_2), (u_3, u_4), $(u_5, u_6), \cdots$ or (u_2, u_3), (u_4, u_5), $(u_6, u_7), \cdots$, the points $t = u_1, u_2, u_3, \cdots$ being both the zeros and the points of inflexion of $x = x(t)$. The properties of being concave or convex should be meant in their strict sense, that is, so as to exclude rectilinear segments. In particular, $x(t)$ is nowhere constant, and so there exists on each of the intervals (u_n, u_{n+1}) a unique point, say $t = v_n$, at which the derivative of $x(t)$ vanishes. The absolute value of $x(v_n)$ then is the maximum of $|x(t)|$ on the "half-wave" (u_n, u_{n+1}). By the "n-th amplitude," a_n, of $x(t)$ will be meant this (local) absolute maximum.

In the above notations,

$$a_n = (-1)^{n+1} x(v_n) > 0, \qquad x'(v_n) = 0, \tag{13}$$

since

$$\operatorname{sgn} x(t) = (-1)^{n+1} \text{ if } u_n < t < u_{n+1}, \tag{14}$$

and

$$u_n < v_n < u_{n+1}. \tag{15}$$

Correspondingly, the absolute value of the slope $x'(t)$ has on the interval (v_n, v_{n+1}) a unique maximum, say b_n, and

$$x(u_n) = 0, \qquad b_n = (-1)^{n+1} x'(u_n) > 0, \tag{16}$$

the zeros $t = u_n$ being precisely the points of inflexion. The (local) absolute maximum, b_n, of $x'(t)$ will be referred to as the "n-th co-amplitude" of $x(t)$.

(i) *Every non-trivial solution $x(t)$ of (5) is quasi-harmonic.*

In (i), and in (ii), (iii), $\cdots$ below, the coefficient function, $\phi(t)$, occurring in (5) is meant to be a real-valued, continuous function satisfying the (L^∞)-condition (6), and $x(t)$ is any real-valued solution (which does not vanish identically).

In view of (6), it can be assumed that $1 + \phi(t)$ is positive on the half-line $t_0 < t < \infty$. Then (5) shows that $x(t)$ and $x''(t)$ have opposite signs, with the understanding that $x(t) = 0$ if and only if $x''(t) = 0$. This means that the curve $x = x(t)$ is turning its concavities toward the t-axis, with the understanding that the points of inflexion and only these points are on the t-axis. And $x(t)$ does not vanish identically; hence, by a standard property of the solutions of any differential equation (3), the roots t of $x(t) = 0$ cannot cluster, and so the same holds for the roots of $x''(t) = 0$. Consequently, the curve $x = x(t)$ cannot contain a rectilinear segment. This proves that the function $x(t)$ has all properties of a quasi-harmonic function, except possibly those requiring the existence of a sequence of zeros $u_1, u_2, \cdots$ and of a positive limit for the distance $u_{n+1} - u_n$.

In order to ascertain for $x(t)$ these properties also, let $\epsilon > 0$ and $t_\epsilon < t < \infty$. Then, if t_ϵ is large enough,

$$1 - \epsilon < 1 + \phi(t) < 1 + \epsilon, \qquad (\epsilon < 1),$$

by (6). Hence, (5) is "surrounded," in the sense of Sturm's oscillation theory, by two linear differential equations (3) the solutions of which are of the form

$$a \cos (t(1 \pm \epsilon)^{\frac{1}{2}} - \alpha),$$

where a and α are integration constants. But (if $a \neq 0$) the distance between two consecutive zeros of either of these harmonics is a function of ϵ, namely $\pi/(1 \pm \epsilon)^{\frac{1}{2}}$, which tends to π as $\epsilon \to 0$. And this limit process can be effected by letting the end-point of the half-line $t_\epsilon < t < \infty$ tend to ∞. It follows therefore from the simplest of Sturm's comparison theorems, that $x(t)$ has a sequence of consecutive zeros, say $u_1, u_2, \cdots$, and that

$$u_{n+1} - u_n \to \pi \text{ as } n \to \infty ; \tag{17}$$

π being the half of the common wave-length of the non-trivial solutions of the approximation $x'' + x = 0$ to (6).

This completes the proof of (i).

4. Somewhat deeper lies the following fact:

(ii) *If a_n denotes the n-th amplitude of a non-trivial solution $x(t)$ of* (5), *then, as* $n \to \infty$,

$$a_{n+1}/a_n \to 1. \tag{18}$$

Moreover, if b_n denotes the n-th co-amplitude of $x(t)$, then

$$a_n/b_n \to 1 \tag{19}$$

(*hence* $b_{n+1}/b_n \to 1$).

If $1 + \phi(t)$ in (5) is replaced by $\omega^2 + \phi(t)$, where ω is a positive constant, it is clear from (13) and (16) that the resulting change in the unit of time amounts to the replacement of 1 by $1/\omega$ on the right of the limit relation (19). In contrast, the 1 on the right of (18) has nothing to do with the choice, $\omega = 1$, of the t-unit in (5).

A proof of (ii) follows by applying a well-known device of Liapunov, rediscovered by G. D. Birkhoff (for references, cf. [8], pp. 424-425). It is the same device of square-sums on which Perron's theory [3] of the formally unstable case of (9) is based. In the case of (5), the square-sum of Liapunov has a simple meaning, since it can be interpreted as the energy, say $h = h(t)$, of the "varied" linear oscillator, calculated under the assumption that the presence of the small disturbance, $\phi(t)$, is neglected in the Hamiltonian function, but not in the solution, $x(t)$, of (5); so that

$$h = \tfrac{1}{2}(x'^2 + x^2). \tag{20}$$

Differentiation of (20), where $x = x(t)$ is the given solution of (5), gives $h' = xx'' + x'^2$. Hence, if x'' is substituted from (5), it follows that $h' = -xx'\phi$. But $|xx'| \leqq h$, by (20). Consequently, $|h'| \leqq h\,|\phi|$. This differential inequality for $h = h(t)$ and $\phi = \phi(t)$ can be written in the form $|(\log h)'| \leqq |\phi|$, since $h \neq 0$. In fact, (20) is positive throughout, since the simultaneous vanishing of x and x' (for some t) leads to that solution $x(t)$ of (5) which vanishes identically, that is, to the solution excluded in (ii).

The last differential inequality is equivalent to the estimate

$$|\log h(\beta)/h(\alpha)| \leqq \int_{\alpha}^{\beta} |\phi(t)|\,dt, \tag{21}$$

in which α ($> t_0$) and β ($> \alpha$) are arbitrary. And (21) readily leads to (ii). In fact, if either $\alpha = v_n$, $\beta = v_{n+1}$ or $\alpha = u_n$, $\beta = v_n$, it is seen from (15),

(17) and (6) that the integral on the right of (21) tends to 0 as $n \to \infty$. It follows therefore from (21) that

$$\log h(v_{n+1})/h(v_n) \to 0 \text{ and } \log h(v_n)/h(u_n) \to 0$$

as $n \to \infty$. But it is seen from (13), (16) and (20) that $h(v_n) = a_n^2$ and $h(u_n) = b_n^2$. Hence, the last formula line proves both assertions, (18) and (19), of (ii).

5. Of the same type as (ii) is the following fact:

(iii) *If c_n denotes the (absolute) area of the n-th half-wave of a non-trivial solution $x(t)$ of (5), then, as $n \to \infty$,*

$$c_{n+1}/c_n \to 1. \tag{22}$$

Moreover,

$$\tfrac{1}{2}c_n/b_n \to 1. \tag{23}$$

First, from (5),

$$x'(\alpha) - x'(\beta) = \int_\alpha^\beta x(t)\,dt + \int_\alpha^\beta x(t)\phi(t)\,dt,$$

where α and β are arbitrary. Hence, if

$$\alpha \to \infty, \qquad 0 < \beta - \alpha = O(1) \text{ and } \operatorname{sgn} x(t) = \text{const. when } \alpha < t < \beta,$$

then, acording to (6),

$$|x'(\alpha) - x'(\beta)| = \left|\int_\alpha^\beta x(t)\,dt\right| + o\left(\int_\alpha^\beta |x(t)|\,dt\right).$$

But (17), (15) and (14) show that the conditions required of α and β for the truth of this o-relation are satisfied if $\alpha = u_n$, $\beta = u_{n+1}$ and $n \to \infty$. And, according to (16), the expression on the left of the last formula line then becomes identical with the sum $b_n + b_{n+1}$, whereas the first integral on the right is precisely the value c_n defined before (22). Finally, the second integral, that following the o-sign, is majorized by $\beta - \alpha$ times the maximum of $|x(t)|$ on the t-interval (α, β) and so, since $(\alpha, \beta) = (u_n, u_{n+1})$, by $u_{n+1} - u_n$ times the n-th amplitude, a_n. Accordingly,

$$b_n + b_{n+1} = c_n + (u_{n+1} - u_n)\, o(a_n)$$

as $n \to \infty$. Hence, (23) follows from (19), (18) and (17). Finally, (19) and (18) show that (22) is a corollary of (23).

The truth of what would correspond to (17) in the same way as $a_{n+1}/a_n \to 1$ corresponds to $b_{n+1}/b_n \to 1$, that is, the truth of $v_{n+1} - v_n \to \pi$ (and even the existence of $\lim (v_{n+1} - v_n)$ as $n \to \infty$) remains undecided. All that will be shown is that

$$0 < \liminf_{n\to\infty} (v_{n+1} - v_n) \leqq \limsup_{n\to\infty} (v_{n+1} - v_n) \leqq \pi. \tag{24}$$

Clearly, the content of the first of the inequalities (24) can be described as follows:

(iv) *As $n \to \infty$, the abscissae of the amplitudes of a non-trivial solution $x(t)$ of (5) cannot cluster at the ends of the respective half-waves.*

This can be interpreted as a limitation of the way in which the uniform continuity of $x(t)$ on a bounded interval $t_0 < t < t^0$ can deteriorate as $t^0 \to \infty$. Needless to say, (i) and (ii) do not imply (and, in view of Perron's counter-example, cannot imply) that $x(t)$ remains bounded during this limit process. All that is true is that, as $t \to \infty$,

$$x(t) = O(1) \text{ if and only if } x'(t) = O(1). \tag{25}$$

In fact, since $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ represent the local maxima of $|x(t)|$ and $|x'(t)|$ respectively, (25) is a corollary of (19).

If $t \to \infty$, then $x''(t) = -x(t) + x(t)o(1)$, by (5) and (6). Since a_n is the maximum of $|x(t)|$ on the n-th half-wave, it follows that $x''(t) = -x(t) + a_n o(1)$ holds uniformly in t if $v_n \leqq t \leqq v_{n+1}$ and $n \to \infty$; cf. (15), (17), and (18). Consequently, by Taylor's formula,

$$x(v_{n+1}) - x(v_n) = x'(v_n)(v_{n+1} - v_n) + \tfrac{1}{2}x''(v_n)(v_{n+1} - v_n)^2 + o(a_n)o(v_{n+1} - v_n)^2.$$

Hence, from (13),

$$a_{n+1} + a_n = \tfrac{1}{2}a_n(v_{n+1} - v_n)^2 + o(a_n)o(v_{n+1} - v_n)^2,$$

since $x''(v_n) = -x(v_n) + o(|x(v_n)|)$, by (5) and (6). If this representation of $a_{n+1} + a_n$ is divided by a_n, it is seen that the assumption that the values $(v_{n+1} - v_n)^2$ can cluster at 0 contradicts the relation (18). This contradiction proves the first, while (15) and (17) imply the last, of the inequalities (24).

6. It turns out that, in a certain sense, (i) and (ii) are of a final nature. For instance, it is clear from the Perron type of example which is defined by

(4) and (8) that, although the amplitude must satisfy (18), not only $a_n \neq O(1)$ but even $a_n \to \infty$ is possible. In the same example, $x(t)$ is seen to be the sum of $t \cos t$ and of a periodic function; so that $x(t)$ is certainly not almost-periodic in Besicovitch's sense.

In contrast, if the (L^∞)-condition, (6), is replaced by the (L^1)-condition, (10), then the n-th amplitude of $x(t)$ must tend to a finite (and non-vanishing) limit, which implies, among other things, that (2) cannot be violated. And $x(t)$ must be almost-periodic even in Weyl's restricted sense. All of this, and more, is contained in the theorem italicized after (9). On the other hand, nothing like (i) or (ii) can be true if (6) is replaced by (10). In order to see this, it is sufficient to observe that (10) can in no sense restrict the behavior of a continuous (or, for that matter, regular-analytic) function $\phi(t)$ on a sequence of t-intervals (t_1^*, t_1^{**}), $(t_2^*, t_2^{**}), \cdots$ satisfying $t_n^{**} - t_n^* \to \infty$ as $n \to \infty$.

The example (4) defined by (8) is of a somewhat primitive nature. In particular, its amplitudes *tend* to infinity. More pathological possibilities compatible with (6) can be realized by applying the following rule of construction:

Let $g(t)$, where $0 \leqq t < \infty$, be a real-valued function possessing a continuous first derivative and satisfying

$$g(t) \to 0 \text{ and } g'(t) \to 0 \text{ as } t \to \infty \tag{26}$$

(neither g nor g' need be monotone). Then, if $G(t)$ denotes the "Fourier primitive,"

$$G(t) = \int_0^t g(s) \cos s \, ds, \tag{27}$$

of $g(t)$, the function

$$x(t) = e^{G(t)} \cos t, \tag{28}$$

is a solution of a differential equation (5) in which $\phi(t)$ is a real-valued continuous function satisfying (6).

This rule supplies examples of various type, since, in view of the parenthetical remark following (26), the assumptions imposed on $g(t)$ can hardly restrict the oscillations of (27) or of the amplitudes, a_n, of (28) (the sequence $a_1, a_2, \cdots$ is that of the relative maxima of the absolute value of (28)).

In order to verify the rule, let $r = r(t)$, where $0 \leqq t < \infty$, be a real-

valued function which is of class C'' and has the following properties: The assignment (7) defines a function $\phi(t)$ which (notwithstanding the $\tan t$) is *continuous,* and such as to satisfy the (L^∞)-condition, (6). As was seen in the introduction, the function (4) defined by any such $r(t)$ is a solution of the differential equation (5) which belongs to the coefficient function (7). Hence, it is sufficient to verify that the conditions required of $r(t)$ are fulfilled by $e^{G(t)}$, if $G(t)$ is the "cosine primitive" of the function $g(t)$ which, in turn, is of class C' and satisfies (26). In fact, the rule can then be verified, just by identifying (4) with (28), as follows:

First, $G' = g \cos t$, by (27). Hence, if $r = e^G$,

$$r' = rg \cos t.$$

This representation of r' implies that

$$r'' = r\,(g^2 \cos^2 t + g' \cos t - g \sin t).$$

But the last two relations reduce (7) to

$$-\phi = g^2 \cos^2 t + g' \cos t + g \sin t;$$

so that the $\tan t$ becomes eliminated. Thus, $g(t)$ being of class C', the function $\phi(t)$ defined by the last formula line is continuous. And it satisfies (6), since (26) is assumed.

In view of the "converse" assertion following (12), no non-trivial solution of (5) can satisfy

(29) $$x(t) \to 0 \text{ as } t \to \infty,$$

if the continuous coefficient function is subject to (10). But if (10) is replaced by (6), it now follows that, *notwithstanding* (i), (ii), (iii), *a non-trivial solution of* (5) *can satisfy* (29). In fact, the preceding verification remains unaltered if $\cos s$, $\cos t$ in (27), (28) are changed to $\sin s$, $\sin t$ respectively. Hence, in order to satisfy (6) and (29), where $x(t) \neq$ const., it is sufficient to ascertain that, for a function $g(t)$ of class C',

(29 bis) $$\int_0^t g(s) \sin s \, ds \to -\infty \text{ as } t \to \infty$$

is not prevented by (27) alone. But this is seen from the example $g(t) = -\sin t/t$.

Since (29) is equivalent to $a_n \to 0$ as $n \to \infty$, this example proves the existence of the second possibility in the following assertion:

(v) *Put*

$$\lambda = \liminf_{n\to\infty} a_n \text{ and } \mu = \limsup_{n\to\infty} a_n, \tag{30}$$

where a_n denotes the n-th amplitude of a real, non-trivial solution $x(t)$ of a differential equation (5) *in which $\phi(t)$ is a continuous function satisfying* (6). *Then each of the seven possibilities*

$$0 < \lambda = \mu < \infty, \qquad \lambda = 0 = \mu, \qquad \lambda = \infty = \mu, \qquad 0 \leqq \lambda < \mu \leqq \infty$$

can actually occur.

The first of these possibilities is realized, of course, by $x'' + x = 0$, where $\phi = 0$. The third possibility is exemplified by (4) and (8). Needless to say, the third case, too, can be subordinated to the above rule. Correspondingly, the four remaining possibilities, that is, the "mixed" cases $0 \leqq \lambda < \mu \leqq \infty$, can be obtained by alternating the choice of $\phi(t)$ on consecutive t-intervals (of appropriately constructed lengths) in such a way that sometimes the second, and sometimes the third (or first), of the three "pure" types preponderates.

In view of (v), there are at most two ways of improving on (i), (ii), one being the adjunction to (6) of a metric assumption, (10), the other the restriction of $\phi(t)$ by some qualitative condition (possibly some restriction of monotony). In the latter regard, it seems to be a general principle that, by virtue of (6) alone, there is a certain *balance* between the asymptotic behaviors of two linearly independent solutions of (5), as illustrated by the following fact:

If (5) *has a* (*non-trivial*) *solution the amplitudes of which tend to* 0, *then* (5) *has another* (*linearly independent*) *solution the amplitudes of which tend to* ∞.

In other words, the occurrence of the second of the possibilities established by (v) necessitates, for one and the same differential equation (5) satisfying (6), the occurrence of the third of the possibilities. In fact, let $x(t)$ and $x^*(t)$ be two linearly independent solutions, and let a_n^*, b_n^*, u_n^*, v_n^* be the constants which belong to $x^*(t)$ in the same way as the constants a_n, b_n, u_n, v_n occurring in (13), (14), (15), (16) belong to $x(t)$. According to (13), the Wronskian of $x(t)$ and $x^*(t)$ at $t = v_n$ is $\pm a_n$ times the derivative of $x^*(t)$ at $t = v_n$. But a Wronskian belonging to any differential equation (3) is independent of t, and its (constant) value is 0 if and only if $x(t)$ and $x^*(t)$ are linearly dependent. Consequently, if $a_n \to \infty$, that is, if $x(t)$ satisfies the assumption, (29), then the absolute value of the derivative of $x^*(t)$

at $t = v_n$ must *tend* to ∞ as $n \to \infty$. Since the co-amplitude, b_n^*, of $x^*(t)$ is the maximum of the absolute value of the derivative of $x^*(t)$ between $t = v_n^*$ and $t = v_{n+1}^*$, it now follows from (15) and (17) that $b_n^* \to \infty$. In view of (18) and (19), this implies the assertion, $a_n^* \to \infty$. A corollary can be formulated as follows:

If (5) *has a* (*non-trivial*) *solution satisfying* (29), *then* (5) *has a* (*linearly independent*) *solution violating* (3).

It is also seen that (29) is equivalent to the corresponding assumption for $x'(t)$, that is,

$$x(t) = o(1) \text{ if and only if } x'(t) = o(1).$$

This corresponds to (25).

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[7] A. Wintner, "On a statement of Fatou," *Journal of the London Mathematical Society*, vol. 11 (1936), pp. 245-246; cf. also "The periodic analytic continuations of the circular orbits in the restricted problem of three bodies," *Proceedings of the National Academy of Sciences*, vol. 22 (1936), pp. 434-439, where further references are given.

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ON COMPONENTS OF A FUNCTION AND ON FOURIER TRANSFORMS.*

By H. KOBER.

1. Introduction. Let $F(\zeta)$ be defined on a Jordan curve C in the complex plane. Various writers have treated the problem of representing $F(\zeta)$ in the form $F(\zeta) = F_1(\zeta) + F_2(\zeta)$; the $F_j(\zeta)$ $[j = 1, 2]$ are required to be the limit-functions of functions $F_j(z)$ $[z = x + iy;\ z \to \zeta,\ \zeta \text{ on } C]$ which are analytic in the interior or exterior of C, respectively.[1]

Now replace C by the real axis. It is known that any function $F(x) \in L_2(-\infty, \infty)$ is representable in the form $F(x) = F_1(x) + F_2(x)$ where $F_1(x)$ or $F_2(x)$ is the limit-function of an element $F_1(z)$ of $\mathfrak{H}_2$ or $F_2(z)$ of $\bar{\mathfrak{H}}_2$ $[z = x + iy;\ y \to 0]$, respectively;[2] the class $\mathfrak{H}_p$ has been introduced by E. Hille and J. D. Tamarkin.[3] A similar result holds for $F(x) \in L_p(-\infty, \infty)$ $[1 < p < \infty]$; also for $p = 1$ and $p = \infty$, though the result is here less simple. There are, however, a number of further cases of interest, first of all the most general case $F(x)(1 + x^2)^{-1} \in L_1(-\infty, \infty)$. The problems thus arising will be treated in the present paper in detail, starting with preliminary results concerning the case where C is the unit circle.

The resolution of a function, defined on a Jordan curve C or on the real axis, into its components is a useful tool for analysis, for it is adapted to the reduction of problems on Lebesgue-integrable functions to problems on analytic functions. In fact, it has been applied in the approximation by rational

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[1] E. g., J. Plemlj, *Monatshefte für Mathematik und Physik*, vol. 19 (1908), pp. 205-210; his method of proof, however, is not strict. J. L. Walsh, *Comptes Rendus*, vol. 178 (1924), pp. 58-59, and "Interpolation and approximation by rational functions in the complex domain," *American Mathematical Society Colloquium Publications*, 1935. A. Ghika, *Comptes Rendus*, vol. 186 (1928), pp. 1808-1810, and vol. 202 (1936), pp. 278-280; his results, and some more general and more detailed ones, can be deduced from **2** of the present paper by means of a conformal transformation.

[2] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford 1937, Theorem 98.

[3] *Fundamenta Mathematicae*, vol. 25 (1935), pp. 329-352. $\mathfrak{H}_p$ is the class of functions $F(z)$ which, for $y > 0$, are regular and satisfy the inequality

$$\int_{-\infty}^{\infty} |F(x + iy)|^p\, dx \leq M^p \quad [0 < p < \infty] \quad \text{or} \quad |F(z)| \leq M \quad [p = \infty],$$

respectively. A function $F(z)$ is said to belong to $\bar{\mathfrak{H}}_p$ when $F(-z)$ belongs to $\mathfrak{H}_p$.

functions [4] and in the solution of integral equations [4] and, implicitly, in the theory of Fourier series. In the present paper the theory is employed to establish necessary and sufficient conditions for a function to be represented as a Fourier or Fourier-Stieltjes transform, using the Widder [5] theory of Laplace transforms. The theory is further used to deal with the Stieltjes transformation (**13-14**) and with Hilbert transforms (**15**), and to generalize the equation $2\cos z = e^{iz} + e^{-iz}$ (**6-7**).

The following notations will be used.[6]

$$(1.1)\quad Hf = H[f;\phi] = \frac{1}{2\pi}\,\text{P. V.}\int_{-\pi}^{\pi} f(e^{i\theta})\cot\tfrac{1}{2}(\phi-\theta)\,d\theta$$

[P. V. = Principal Value].

$$(1.2)\quad \mathfrak{H}F = \mathfrak{H}[F(t);x] = \frac{1}{\pi}\,\text{P. V.}\int_{-\infty}^{\infty}\frac{F(t)\,dt}{t-x},$$

$$\mathfrak{R}F = \mathfrak{R}[F(t);x] = \frac{1}{\pi}\,\text{P. V.}\int_{-\infty}^{\infty} F(t)\left(\frac{1}{t-x} - \frac{t}{t^2+1}\right)dt.$$

$$(1.3)\quad f_j(z) = \frac{(-1)^{j-1}}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)\,d\zeta}{\zeta - z} \qquad \left[\begin{matrix}|z|<1\\ |z|>1\end{matrix}\ \text{for}\ \begin{matrix}j=1\\ j=2\end{matrix}\right]$$

where the integral is taken over the unit circle, and

$$(1.41)\qquad F_j(z) = \frac{(-1)^{j-1}}{2\pi i}\int_{-\infty}^{\infty}\frac{F(t)\,dt}{t-z},$$

$$(1.42)\qquad \Phi_j(z) = \frac{(-1)^{j-1}}{2\pi i}\int_{-\infty}^{\infty} F(t)\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)dt,$$

where $z = x + iy$, and $y > 0$ for $j = 1$, $y < 0$ for $j = 2$; $\mathfrak{H}F$ is the Hilbert operator, and evidently

$$\mathfrak{H}F = \mathfrak{R}F + C_0, \qquad C_0 = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{tF(t)}{t^2+1}\,dt$$

whenever the latter integral and either $\mathfrak{H}F$ or $\mathfrak{R}F$ exist. Similarly

$$F_j(z) = \Phi_j(z) - (-1)^j C_0.$$

Constants depending on λ only are denoted by the single symbol A_λ.

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[4] J. L. Walsh, *loc. cit.*, and H. Kober, *Proceedings of the Edinburgh Mathematical Society*, (2) vol. 7 (1946), pp. 123-133. I. *Vecoua, C. R. (Doklady) de l'Académie des Sci. de l'URSS*, vol. 26 (1940), 134, pp. 327-330.

[5] D. V. Widder, *The Laplace Transform*, Princeton, 1941.

[6] For the operators Hf and $\mathfrak{H}F$, see M. Riesz, *Mathematische Zeitschrift*, vol. 27 (1928), pp. 218-244. For $\mathfrak{R}F$, see H. Kober, *Journal of the London Mathematical Society*, vol. 18 (1943), pp. 66-71; there in the first footnote, p. 69, the "(3.2)" is to be omitted, the correct result is given in Lemma 11 of the present paper.

2. On functions defined on the unit circle. Except for Lemma 2, the results of this section are either known or, in substance, equivalent to known theorems on Fourier series.

LEMMA 1. *Let* $1 < p < \infty$ *and* $f(e^{i\theta}) \in L_p(-\pi, \pi)$. *Then*

$$(2.1)\quad \int_{-\pi}^{\pi} |f_j(re^{i\theta})|^p d\theta \leq A_p^p \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta \qquad \begin{bmatrix} 0 < r < 1 & \text{for } j = 1 \\ r > 1 & \text{for } j = 2 \end{bmatrix}.$$

LEMMA 2. *Let* $f(e^{i\theta}) \in L_1(-\pi, \pi)$ *and let* $0 < \lambda < 1$, $\nu < 1 - \lambda$. *Then*

$$(2.2)\quad \left(\int_{-\pi}^{\pi} \frac{|f_j(re^{i\theta})|^\lambda d\theta}{|1 + re^{i\theta}|^\nu}\right)^{1/\lambda} \leq A_{\lambda,\nu} \int_{-\pi}^{\pi} |f(e^{i\theta})|\, d\theta \begin{bmatrix} 0 < r < 1 & \text{for } j = 1 \\ r > 1 & \text{for } j = 2 \end{bmatrix}.$$

Lemma 1 is deduced from the inequality

$$\int_{-\pi}^{\pi} d\phi \left| \int_{-\pi}^{\pi} g(\theta) \frac{e^{i\theta} + re^{i\phi}}{e^{i\theta} - re^{i\phi}} d\theta \right|^p \leq A_p^p \int_{-\pi}^{\pi} |g(\theta)|^p d\theta \qquad \begin{bmatrix} 0 < r < 1 \\ 1 < p < \infty \end{bmatrix},$$

due to M. Riesz, taking $g(\theta) = f(e^{i\theta})$ and observing that

$$\int_{-\pi}^{\pi} g(\theta) \frac{e^{i\theta} + re^{i\phi}}{e^{i\theta} - re^{i\phi}} d\theta = -\int_{-\pi}^{\pi} f(e^{i\theta}) d\theta + 4\pi f_1(re^{i\phi}).$$

Evidently (2.1) holds for any p $[1 < p < \infty]$ if $f(e^{i\theta})$ belongs to $L_\infty(-\pi, \pi)$.

For $\nu = 0$ and $j = 1$ $[r < 1]$, (2.2) is in substance due to J. E. Littlewood.[7] From this result, (2.2) follows for $\nu = 0$ and $j = 2$ $[r > 1]$ by the identity

$$f_2(z) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) d\theta + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta^{-1}) d\zeta}{\zeta - z^{-1}} \qquad [|z| > 1].$$

To prove the general case we fix a number $q > 1$ such that $1 - \lambda > q^{-1} > \nu$ and take $\nu > 0$, $s = q(q-1)^{-1}$. By Hölder's inequality, the left side of (2.2) is not greater than

$$\left\{ \left(\int_{-\pi}^{\pi} |f_j(re^{i\theta})|^{\lambda s} d\theta \right)^{1/s} \left(\int_{-\pi}^{\pi} \frac{d\theta}{|1 + re^{i\theta}|^{\nu q}} \right)^{1/q} \right\}^{1/\lambda} \leq A_{q,\lambda} \left(\int_{-\pi}^{\pi} |f_j(re^{i\theta})|^{\lambda s} d\theta \right)^{1/(\lambda s)},$$

since $\nu q < 1$. We have $0 < \lambda s < 1$; applying the Littlewood result, we complete the proof.

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[7] *Journal of the London Mathematical Society*, vol. 1 (1926), pp. 229-231.

Let $f(e^{i\theta}) \in L_1(-\pi, \pi)$. *Then, for almost all* ϕ *in* $(-\pi, \pi)$,

$$(2.3)\quad f_j(z) \to \tfrac{1}{2} f(e^{i\phi}) - \frac{i}{2}(-1)^j \{H[f;\phi] + \frac{1}{2\pi i}\int_{-\pi}^{\pi} f(e^{i\theta})\, d\theta\} \begin{bmatrix} z = re^{i\phi} \\ r \to 1 \end{bmatrix}.$$

This result is known. From the lemmas and from (2.3), we deduce

THEOREM 1. *Let* $1 \leq p \leq \infty$ *and* $f(e^{i\theta}) \in L_p(-\pi, \pi)$. *Then* $f(e^{i\theta})$ *can be represented in the form* $f(e^{i\theta}) = f_1(e^{i\theta}) + f_2(e^{i\theta})$ *where* $f_j(e^{i\theta})$ $[j = 1, 2]$ *is the limit-function of* $f_j(z)$ *as* $z \to e^{i\theta}$. *When* $1 < p < \infty$ *then the functions* $f_1(z)$ *and* $f_2(z^{-1})$ *belong to the Riesz class* H_p.[8] *If* $p = 1$ *or* $p = \infty$, *then they belong to* H_q *for any positive* q *smaller than* p, *while they belong also to* H_p *if and only if* Hf *belongs to* $L_p(-\pi, \pi)$.

The last assertion is deduced by means of Fatou's theorem and of the Smirnoff theorem:[9] If $0 < p < P \leq \infty$ and $g(z) \in H_p$, and if $g(e^{i\theta})$, the limit-function of $g(z)$, belongs to $L_P(-\pi, \pi)$, then $g(z) \in H_P$.

3. On functions defined on the real axis.

THEOREM 2. *Let* $F(x)(1+x^2)^{-1} \in L_1(-\infty, \infty)$. *Then* $F(x)$ *can be represented in the form* $F(x) = F_1(x) + F_2(x)$ *where* $F_j(x)$ $[j = 1, 2]$ *is the limit function of* $\Phi_j(z)$ [*see* 1.4]. *Again*

$$(3.1)\quad \left\{\int_{-\infty}^{\infty} \frac{|\Phi_j(x+iy)|^\lambda dx}{\{x^2 + (1+|y|)^2\}^\mu}\right\}^{1/\lambda} \leq A_{\lambda,\mu} \int_{-\infty}^{\infty} \frac{|F(t)|\, dt}{1+t^2} \begin{bmatrix} y > 0 \\ y < 0 \end{bmatrix} \text{ for } \begin{matrix} j = 1 \\ j = 2 \end{matrix}$$

for $0 < \lambda < 1$, $2\mu > 1 + \lambda$, *uniformly with respect to* y.

Some better results can be proved for the following particular cases:

(A) $F(x)(1+|x|)^{-1} \in L_1(-\infty, \infty)$. (B) $F(x) \in L_1(-\infty, \infty)$.

(C) $F(x) \in L_p(-\infty, \infty)$, $1 < p < \infty$. (D) $F(x) \in L_\infty(-\infty, \infty)$.

It will be shown that $F_j(x)$ is the limit function of $F_j(z)$ or $\Phi_j(z)$, respectively, where $y > 0$ for $j = 1$; $y < 0$ for $j = 2$;

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[8] F. Riesz, *Mathematische Zeitschrift*, vol. 18 (1923), pp. 87-95. A function $f(z)$ belongs to H_p if, for $|z| < 1$, it is regular and satisfies the inequality

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \leq M^p \quad [0 < r < 1]$$

when $0 < p < \infty$; $|f(z)| < M$ when $p = \infty$. For the connection between the classes $\mathfrak{H}_p$ and H_p, see Hille-Tamarkin, *loc. cit.*, and H. Kober, *Bulletin of the American Mathematical Society*, vol. 49 (1943), pp. 437-443, Lemma 2.

[9] E. g., A. Zygmund, *Trigonometric Series*, Warsaw, 1935, 7.56 (iv).

$$(3A)\quad \left(\int_{-\infty}^{\infty}\frac{|F_j(x+iy)|^\lambda dx}{\{x^2+(1+|y|)^2\}^\mu}\right)^{1/\lambda}\leq A_{\lambda,\mu}\int_{-\infty}^{\infty}\frac{|F(t)|\,dt}{1+|t|}\quad\left[\begin{matrix}0<\lambda<1,\\ \mu>\frac{1}{2}\end{matrix}\right];$$

$$(3B)\quad \left(\int_{-\infty}^{\infty}\frac{|F_j(x+iy)|^\lambda dx}{\{x^2+(1+|y|)^2\}^\mu}\right)^{1/\lambda}\leq A_{\lambda,\mu}\int_{-\infty}^{\infty}|F(t)|dt\quad\left[\begin{matrix}0<\lambda<1,\\ 2\mu>1-\lambda\end{matrix}\right];$$

$$(3C)\quad \int_{-\infty}^{\infty}|F_j(x+iy)|^p dx\leq A_p{}^p\int_{-\infty}^{\infty}|F(t)|^p dt\qquad[1<p<\infty];$$

$$(3D)\quad \int_{-\infty}^{\infty}\frac{|\Phi_j(x+iy)|^q dx}{\{x^2+(1+|y|)^2\}^\mu}<\infty\qquad\left[\begin{matrix}1<q<\infty,\\ \mu>\frac{1}{2}\end{matrix}\right],$$

$$\leq A_q{}^q\int_{-\infty}^{\infty}\frac{|F(t)|^q dt}{1+t^2}\qquad\left[\begin{matrix}1<q<\infty,\\ \mu\geq 1\end{matrix}\right];$$

and the following theorems will be deduced:

THEOREM 2′. *Let $F(t)\in L_p(-\infty,\infty)$. When $1<p<\infty$ then $F_1(z)\in\mathfrak{H}_p$, $F_2(z)\in\bar{\mathfrak{H}}_p$. When $p=1$ then $F_1(z)\in\mathfrak{H}_p$ if and only if $\mathfrak{H}F\in L_1(-\infty,\infty)$; when $p=\infty$ then $\Phi_1(z)\ \mathfrak{H}_p$ if and only if $\mathfrak{R}F\in L_\infty(-\infty,\infty)$; the conditions $F_1(z)\in\mathfrak{H}_1$ and $F_2(-z)\in\mathfrak{H}_1$ are equivalent and so are the conditions $\Phi_1(z)\in\Phi_2(-z)\in\mathfrak{H}_\infty$.*

THEOREM 2″ (*Uniqueness theorem*). (a) *Whenever a function $F(x)$ is represented in the form $F(x)=\bar{F}_1(x)+\bar{F}_2(x)$ where $\bar{F}_j(z)$ is the limit-function of $F_j(z)$ $[j=1,2]$ and $\bar{F}_1(z)\in\mathfrak{H}_p$, $\bar{F}_2(-z)\in\mathfrak{H}_q$, $1\leq p\leq\infty$, $1\leq q\leq\infty$, then this representation is unique, except for a constant when $p=q=\infty$; in the latter case the function $\bar{F}_j(z)$ is equal to $\Phi_j(z)$ [see 1.42] except for a constant.* (b) *When $0<p<1$ and $0<q<1$, or when it is merely required that $\bar{F}_1(z)$ and $F_2(z)$ shall be regular for $y>0$ or $y<0$, respectively, then the representation is not unique.*

The last statement is evident. For the function $\{(z-1)(z-2)\cdots(z-n)\}^{-1}$ belongs both to $\mathfrak{H}_\lambda$ and $\bar{\mathfrak{H}}_\lambda$ whenever $n^{-1}<\lambda<1$.

Remark[10] *to Theorem* 2. Let (i) $F(x)$ and (ii) $\mathfrak{R}F$ be of bounded variation over $(-\infty,\infty)$. Then $\Phi'_1(z)=d\Phi_1(z)/dz$ belongs to $\mathfrak{H}_1$, $\Phi'_2(z)$ to $\bar{\mathfrak{H}}_1$, and $F(x)$ and $\mathfrak{R}F$ are absolutely continuous in $(-\infty,\infty)$; so is $F_j(x)$, being the integral of $\Phi'_j(x)=\lim\Phi'_j(z)$ $[y\to 0;\ j=1,2]$.

Remark to Theorem 2′. Let (i) $F(x)\in L_1$ and (ii) $\int_{-\infty}^{\infty}F(x)dx=0$.

[10] The proof is rather complicated. The statement can be considered as a generalisation of the Hille-Tamarkin Theorem 3.2 (i and iii).

Then there are functions $F_n(x)$ $[n = 3, 4, 5, \cdots]$ such that both F_n and $\mathfrak{H}F_n$ belong to L_1 and that $\int_{-\infty}^{\infty} |F(x) - F_n(x)|\, dx \to 0$ as $n \to \infty$. The conditions are necessary.[11]

Further remark. The inequalities (3.1), (3A), (3B), (3D) do not hold whenever $2\mu \leq 1 + \lambda$, $2\mu \leq 1$, $2\mu \leq 1 - \lambda$ or $2\mu \leq 1$, respectively.

To show this, take $F(t) = t^2\mathfrak{e}(t)$, $t\mathfrak{e}(t)$, $\mathfrak{e}(t)$ or $\mathfrak{e}(t)$, where $\mathfrak{e}(t) = 1$ for $0 < t < 1$, $\mathfrak{e}(t) = 0$ otherwise.

4. Proof of Theorem 2. By the transformation

$$(4.1)\quad t = \tan \tfrac{1}{2}\theta,\ x = \tan \tfrac{1}{2}\phi,\ z = i(1-w)(1+w)^{-1},\ F(t) = f(e^{i\theta}) \qquad [-\pi < \begin{matrix}\theta\\ \phi\end{matrix} < \pi]$$

we have

$$(4.2)\quad f(e^{i\theta}) \in L_1(-\pi, \pi); \qquad \Re[F(t); x] = -H[f(e^{i\theta}); \phi].$$

$$(4.3)\quad \Phi_j(z) = f_j(w) + \tfrac{1}{2}(-1)^j a_0; \quad a_0 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F(t)\,dt}{t^2+1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\,d\theta.$$

From (2.3), (4.2) and (4.3) we deduce that, for almost all x in $(-\infty, \infty)$,

$$(4.4)\quad \Phi_j(z) \to \tfrac{1}{2}F(x) + (i/2)(-1)^j \Re F \text{ as } y \to 0 \qquad [(-1)^j y < 0].$$

Thus $2F_j(x) = F(x) + (-1)^j i\Re F$, and so $F(x) = F_1(x) + F_2(x)$.

By a known argument, based upon a result due R. M. Gabriel,[12] we can now deduce (3.1) from (2.2), taking $\mu = 1 - \frac{1}{2}\nu$. In a similar way we prove (3D); when $\mu \geq 1$ we use (2.1), while for $\frac{1}{2} < \mu < 1$ we use the inequality

$$\int_{-\pi}^{\pi} \frac{|f_1(re^{i\theta})|^q}{|1 + re^{i\theta}|^\nu}\, d\theta \leq \Big(\int_{-\pi}^{\pi} |f_1(re^{i\theta})|^{qs} d\theta\Big)^{1/s} \Big(\int_{-\pi}^{\pi} |1 + re^{i\theta}|^{-\nu l} d\theta\Big)^{1/l}$$

$$\leq A_{q,r,s}\Big(\int_{-\pi}^{\pi} |f(e^{i\theta})|^{qs} d\theta\Big)^{1/s} < \infty,$$

where

$$f(e^{i\theta}) \in L_\infty(-\pi, \pi);\ 0 < \nu = 2(1-\mu) < 1;\ 0 < 1 - l^{-1} = s^{-1} < 1 - \nu;\ q > 1.$$

To deduce (3A) from (2.2), we employ the transformation

$$(4.5)\quad t, x, z \text{ as in } (4.1);\ F(t) = (1 + e^{i\theta})f(e^{i\theta});\ F_j(z) = (1+w)f_j(w);$$

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[11] H. Kober, *Bulletin of the American Mathematical Society*, vol. 48 (1942), pp. 421-426, Theorem 3.

[12] *Journal of the London Mathematical Society*, vol. 5 (1930), pp. 129-131. Cf. Hille-Tamarkin, proof of Lemmas 2.1 and 2.5, *loc. cit.*

$$\mathfrak{H}[F;x] = (1+e^{i\phi})(ia_0 - Hf);\quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\,d\theta;\quad \nu+\lambda = 2-2\mu.$$

To prove (3B), we replace (4.5) by

(4.6) t, x, z as in (4.1); $\quad F(t) = (1+e^{i\theta})^2 f(e^{i\theta})$;

$$F_j(z) = (1+w)^2 f_j(w) + a_{-1}(1+w);$$

$$\mathfrak{H}F = (1+e^{i\phi})^2(ia_0 - Hf) + 2ia_{-1}(1+e^{i\phi});$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta\ [n=0,-1];\qquad \nu+2\lambda = 2-2\mu.$$

Some further results can be proved, for instance

THEOREM 2‴. *Let* $(1+t^2)^{-1}F(t)$ *or* $(1+|t|)^{-1}F(t)$ *belong to* L_1. *Then* $(i+z)^{-2}\Phi_1(z)$ *or* $(i+z)^{-1}F_1(z)$ *belongs to* $\mathfrak{H}_1$ *if and only if* $(1+x^2)^{-1}\mathfrak{R}F$ *or* $(1+|x|)^{-1}\mathfrak{H}F$, *respectively, belongs to* L_1, *and the condition* $(i+z)^{-2}\Phi_1(z) \in \mathfrak{H}_1$ *or* $(i+z)^{-1}F_1(z) \in \mathfrak{H}_1$ *is equivalent to* $(i+z)^{-2}\Phi_2(-z) \in \mathfrak{H}_1$ *or* $(i+z)^{-1}F_2(-z) \in \mathfrak{H}_1$, *respectively.*

Both in the cases (3A) and (3B) etc. we have, for almost all x in $(-\infty, \infty)$,

$$F_j(z) \to F_j(x) = \tfrac{1}{2}F(x) + (i/2)(-1)^j\mathfrak{H}F \qquad [y \to 0]. \tag{4.7}$$

5. The inequality (3C) is, in substance, due to M. Riesz.[13] Together with (4.7), it yields the first part of Theorem 2′. For the proof of the second part, we need

LEMMA 3. *Let* $p=1$ *or* $p=\infty$ *and* $F(t) \in L_p$. *Then* $F_1(z) \in \mathfrak{H}_1$ *or* $\Phi_1(z) \in \mathfrak{H}_\infty$, *respectively, if and only if* $\mathfrak{H}F \in L_1$ *or* $\mathfrak{R}F \in L_\infty$.

Take first $p=1$. The necessity of the condition follows from (4.7) by Fatou's theorem. Conversely, let both F and $\mathfrak{H}F$ belong to L_1. Then the constant $a_{-1} = (4\pi)^{-1}\int_{-\infty}^{\infty} F(t)\,dt$ vanishes. Hence, by (4.6), both $f(e^{i\theta})$ and Hf belong to $L_1(-\pi,\pi)$. By the last assertion of Theorem 1, therefore, $f_1(w)$ belongs to the Riesz class H_1. By (4.6), we have $f_1(w)(1+w)^2 = F_1(z)$; employing a known result, we have $F_1(z) \in \mathfrak{H}_1$. The case $p=\infty$ is treated in a similar way, using (4.1)-(4.4).

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[13] M. Riesz, *loc. cit.* The first part ($p=1$) of Lemma 3 is proved by the author in the paper cited in [11]; but that method fails for $p=\infty$.

The proof of Theorem 2″ is based on known properties [14] of the "proper Cauchy integral" of an element of $\mathfrak{H}_p$ (cf. Lemma 9, **15**).

COROLLARY. *If $F(x)$ is represented in the form $F(x) = F_1(x) + F_2(x)$ where $F_1(x)$ and $F_2(-x)$ are limit functions of elements of $\mathfrak{H}_\infty$, then both $F(x)$ and $\mathfrak{R}F$ belong to $\mathfrak{H}_\infty$.*

Finally we state the following result:

If (i) $(1+|x|^m)^{-1}F(x) \in L_\infty(-\infty, \infty)$ $[m = 0, \text{or} = 1, \text{or} = 2, \cdots]$ *and* (ii) $\mathfrak{R}[(t+i)^{-m}F(t); x] \in L_\infty(-\infty, \infty)$, *then* $F(x) = F_1(x) + F_2(x)$ *where $F_j(z)$ is analytic for $y > 0$ $[j = 1]$ or $y < 0$ $[j = 2]$, respectively, and $|F_j(z)| < A(1+|z|)^m$ $[(-1)^{j-1}y > 0]$. The conditions are necessary; the components of $F(x)$ are unique except for an arbitrary polynomial of degree not greater than m.*

6. Let $G^{(\rho)}$ be the set of integral functions which are of order not greater than ρ, let $G_\alpha^{(\rho)}$ $[0 < \alpha < \infty]$ be the set of integral functions $F(z)$ such that, given any $\epsilon > 0$, $|F(z)| < A_{F,\epsilon} \exp\{(\alpha+\epsilon)|z|^\rho\}$. We shall prove

LEMMA 4. *Let $F(z) \in G^{(\rho)}$ or $\in G_\alpha^{(\rho)}$, respectively, let $F(t) \in L_p$, $1 \leq p \leq \infty$. Then $F_j(z)$ belongs to $G^{(\rho)}$ or $G_\alpha^{(\rho)}$ $[1 \leq p < \infty]$ and so does $\Phi_j(z)$ $[p = \infty]$; $[j = 1, 2]$.*

Let $1 < p < \infty$ and $F(z) \in G_\alpha^{(\rho)}$, and let C_r be the semi-circle $w = re^{i\theta}$, $-\pi \leq \theta \leq 0$. Then, for $y > 0$, we have

$$2\pi i F_1(z) = \int_{C_r} \frac{F(w)\,dw}{w-z} + \left(\int_{-\infty}^{-r} + \int_r^{\infty}\right) \frac{F(t)\,dt}{t-z} = I_1 + I_2 + I_3.$$

Since r is arbitrary, $F_1(z)$ can be continued analytically over the entire plane and is, therefore, an integral function. We have to show that, given $\delta > 0$, $|F_1(z)| < \exp\{(\alpha+\delta)|z|^\rho\}$. Let $\epsilon > 0$ and $(\alpha+\epsilon)(1+\epsilon)^\rho = \alpha + \delta$; we have $|F(w)| < A_{F,\epsilon} \exp\{(\alpha+\epsilon)|w|^\rho\}$. Let z be fixed, $|z| \geq 1$, and let $r = (1+\epsilon)|z|$. Then

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[14] For $1 \leq p < \infty$ see Hille-Tamarkin, Theorem 2.1 (ii). For $p = \infty$ we define the "proper Cauchy integral" of a function $g(z)$ of $\mathfrak{H}_\infty$ by

$$G(z) = (2\pi i)^{-1} \int_{-\infty}^{\infty} g(t)[1/(t-z) - t/(t^2+1)]\,dt.$$

Then we have $G(z) = g(z) - \frac{1}{2}g(i)$ or $= -\frac{1}{2}g(i)$ for $y > 0$ or $y < 0$, respectively.

$$|I_1| \leq \pi(1+\epsilon^{-1})A_{F,\epsilon}\exp\{(\alpha+\delta)|z|^\rho\},$$

$$|I_j| \leq \left(\int_{-\infty}^{\infty}|F(t)|^p dt\right)^{1/p}\left(\int_r^\infty \frac{dt}{(t-|z|)^{p'}}\right)^{1/p'} < A_{F,\epsilon}\epsilon^{-1/p} \quad \begin{bmatrix} j=2,3 \\ p'=p(p-1)^{-1} \end{bmatrix}$$

Hence $F_1(z) \in G_a^{(\rho)}$. The proof of the remaining assertions is similar.

7. We need not enlarge by formulating the theorems for $G^{(\rho)}$ and $G_a^{(\rho)}$ resulting from the lemma and the preceding theorems. We can, however, deduce some more detailed results for the class $G_a^{(1)}$.

THEOREM 3. *Let $F(z) \in G_a^{(1)}$ and $F(t) \in L_p$.* (a) *When $1 \leq p < \infty$ then*

$$(7.1) \qquad F(z) = F_1(z) + F_2(z); \quad F_j(z) \in G_a^{(1)}; \quad F_1(z) \in \mathfrak{H}_q, \; F_2(-z) \in \mathfrak{H}_q$$

for any q such that $p \leq q \leq \infty$ when $p > 1$, and such that $p < q \leq \infty$ when $p = 1$.

(b) *When $F(t)\log(t) \in L_1$ and $\int_{-\infty}^{\infty} F(t)dt = 0$, then (7.1) holds for $q = 1$.*

(c) *When $p = \infty$, then $F_1(z)$ is bounded for $y > 0$, $F_2(z)$ for $y < 0$, if and only if the function*

$$(7.2) \qquad \Psi(x) = \int_{-\infty}^{\infty} F(t)\left(\frac{t}{t^2+1} - \frac{t-x}{(t-x)^2+1}\right)dt$$

is bounded in $(-\infty, \infty)$. This condition is certainly satisfied if

$$(7.3) \qquad \int_0^x F(t)dt = Ax + O(1) \qquad [x \to \pm\infty].$$

The first assertion follows from Theorem 2′, using Lemma 4 and the Plancherel-Pólya theorem: Let $F(z) \in G_a^{(1)}$ and $F(t) \in L_p$; then, for $p \leq q \leq \infty$, $F(t)$ belongs to L_q.[15] The hypotheses of Theorem 3(b) imply that $\mathfrak{H}F \in L_1$.[16] Using Lemma 3 we deduce (b). Now we have

$$\mathfrak{R}F + \frac{1}{\pi}\Psi(x) = \frac{1}{\pi}\int_0^\infty \{F(x+t) - F(x-t)\}\left(\frac{1}{t} - \frac{t}{t^2+1}\right)dt.$$

Let A_F be the upper bound of $F(x)$ in $(-\infty, \infty)$. Then $|F'(x)| \leq \alpha A_F$,[15] and so

$$|\mathfrak{R}F + \frac{1}{\pi}\Psi(x)| \leq \frac{2}{\pi}\alpha A_F \int_0^\infty \left(\frac{1}{t} - \frac{t}{t^2+1}\right)tdt = \alpha A_F \; [-\infty < x < \infty].$$

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[15] *Commentarii Math. Helv.*, vol. 10 (1937/38), pp. 110-163, § 30.

[16] H. Kober, *Journal of the London Mathematical Society*, vol. 18 (1943), p. 69.

Consequently the boundedness of $\Re F$ in $(-\infty, \infty)$ is equivalent to that of $\Psi(x)$. This gives the first part of (3c). Since $F(x)$ is bounded in $(-\infty, \infty)$, $\Psi(x)$ can be put into the form

$$\Psi(x) = -\lim_{M\to\infty} \int_{-M}^{M} \frac{F(x+t) - F(t)}{t^2+1}\, tdt.$$

Integrating by parts and using the hypothesis (7.3), we have

$$|\Psi(x)| = \Big| \int_{-\infty}^{\infty} dt\, \frac{t^2-1}{(t^2+1)^2} \times \Big\{ \int_0^{x+t} - \int_0^x - \int_0^t \Big\} F(u)\, du \Big| \leq \bar{A}_F \int_{-\infty}^{\infty} \frac{|t^2-1|}{(t^2+1)^2}\, dt.$$

Hence (7.3) implies that $\Psi(x)$ is bounded, which completes the proof.

It can be shown that Theorem 3(c) holds when (7.3) is replaced by

$$F(x) - Ax = \sum_{j=1}^{n} a_j x^{r_j} (\log x)^{\beta_j} + O(1) \quad \text{or} \tag{7.31}$$

$$\sum_{j=1}^{m} b_j |x|^{\tau_j} (\log |x|)^{\sigma_j} + O(1)$$

for $x \to \infty$ or $x \to -\infty$, respectively, where $0 \leq r_j < 1$, $0 \leq \tau_j < 1$; β_j, σ_j real.

The function $z^{-2}(1 - 2\cos z + \cos 2z)$ is an example for the case $1 \leq p = q \leq \infty$, while $\cos z$ is an example for the case $p = q = \infty$, with $A = 0$. The function $F_0(z) = \int_0^z u^{-1} \sin u\, du$, however, belonging to $G_1^{(1)}$, is bounded in $(-\infty, \infty)$, while the functions

$$\Psi(x) = -\int_{-\infty}^{\infty} \frac{tdt}{t^2+1} \int_t^{t+x} \frac{\sin u}{u}\, du = \frac{\pi}{2} \log |x| + O(1), \tag{7.4}$$

$$\Re F_0 = \int_0^x \frac{\cos u - 1}{u}\, du + \text{constant}$$

are not bounded; hence $F_0(z)$ cannot be re represented in the form (7.1), with $q = \infty$. Finally we note that the function

$$F(z) = (\tfrac{1}{4} + z^2)\Gamma(\tfrac{1}{4} + iz/2)\pi^{-\frac{1}{4}-\frac{1}{2}iz}\zeta(\tfrac{1}{2} + iz) \quad [\zeta(s) = \sum_1^{\infty} n^{-s}]$$

is an example for the class $G^{(1)}$ such that $F_1(z) \in \mathfrak{H}_p$, $F_2(z) \in \bar{\mathfrak{H}}_p$ whenever $1 < p \leq \infty$.

8. In the following sections we shall treat the *Fourier-Stieltjes* transform

$$(8.1) \qquad F(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-itx} d\alpha(t),$$

and the *Fourier transform* of a function $f(t)$ of $L_1(-\infty, \infty)$

$$(8.2) \qquad F(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t) e^{-itx} dt$$

and the $\mathfrak{M}^q$ Fourier transform of a function $f(t)$ of $L_p(-\infty, \infty)$ $[1/p + 1/q = 1, 1 < p \leq 2]$

$$(8.3) \quad F(x) = (2\pi)^{-\frac{1}{2}} \mathfrak{M}^q \int_{-\infty}^{\infty} f(t) e^{-itx} dt = (2\pi)^{-\frac{1}{2}} \operatorname*{lim.\,i.\,mean}_{N\to\infty}{}^{\text{order } q} \int_{-N}^{N} f(t) e^{-itx} dt.$$

In (8.1), $\alpha(t)$ is required to be a function of bounded variation over $(-\infty, \infty)$ and is supposed to be normalized, i.e. $\alpha(t) = \frac{1}{2}\{\alpha(t+0) + \alpha(t-0)\}$, $\alpha(0) = 0$. The following conditions are certainly necessary for a function $F(x)$ to be representable as a Fourier-Stieltjes transform: (i) The function

$$(8.4) \qquad h(t) = (2\pi)^{\frac{1}{2}} \lim_{M\to\infty} \int_{-M}^{M} F(x) e^{itx} dx$$

which should, in fact, be equal to $\alpha(t+0) - \alpha(t-0)$, exists for all t in $(-\infty, \infty)$, vanishes except at a finite or enumerable set $\{t_n\}$, and $\sum |h(t_n)| < \infty$, (ii) $F(x)$ is bounded and uniformly continuous in $(-\infty, \infty)$, and (iii) so is $\Re F$. For, by the previous theory, (8.1) implies that

$$i(2\pi)^{\frac{1}{2}} \Re F = \Big(\int_0^{\infty} - \int_{-\infty}^{0}\Big) e^{-itx} d\alpha(t) + A\,; A = \int_0^{\infty} e^{-t} d\{\alpha(t) + \alpha(-t)\}.$$

We note that (ii) implies the existence of functions $f_n(x)$ $[n = 1, 2, \cdots]$ represented as Fourier-Stieltjes transforms, even as Fourier transforms, uniformly bounded with respect to x and n, and such that $f_n(x) \to F(x)$ $[n \to \infty]$ for all x.[17]

These conditions, however, are not sufficient. The function $F_0(x) = \int_0^x u^{-1} \sin u du$ (see **7**) satisfies (i) and (ii), but not (iii) (see 7.4). Consequently it is not a Fourier-Stieltjes transform; the same is, therefore, true for $G(x) = e^{ix} F_0(x)$. Now $G(x)$ satisfies (i) and (ii). Since $G(z) \in \mathfrak{H}_\infty$, and since this implies that $\Re G = iG(x) - iG(i)$ [see 4.4 and [14]], the function $G(x)$ satisfies also (iii).

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[17] Compare this with a fundamental property of Fourier-Stieltjes transforms in the case when both $\alpha(t)$ and the $\alpha_n(t)$ are required to be real monotonous functions. E.g. S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig 1932, Theorem 21.

9. To solve the problem thus arising for *Fourier-Stieltjes transforms* and, correspondingly, for *Fourier transforms*, we have to use the Widder theory of Laplace transforms. Introducing the operator

$$(9.1) \quad g_k(t) = g_k[F(x); t] = \int_{-\infty}^{\infty} \frac{F(x)\,dx}{(1 - itx/k)^{k+1}} \qquad [k = 1, 2, 3, \cdots],$$

we obtain the following results.

THEOREM 4. *The function $F(x)$ is, for almost all x in $(-\infty, \infty)$, equal to* (A) *a Fourier-Stieltjes transform, or* (B) *to the sum of a constant and of the Fourier transform of an element of $L_p(-\infty, \infty)$ $[1 < p \leq 2]$, respectively, if and only if*

$$(9.2) \quad \text{(A)} \int_{-\infty}^{\infty} |g_k(t)|\,dt < M \quad \text{or} \quad \text{(B)} \int_{-\infty}^{\infty} |g_k(t)|^p dt < M^p \quad [k = 1, 2, \cdots]$$

where M is independent of k.

THEOREM 4′. *The function $F(x)$ is, for almost all x in $(-\infty, \infty)$, equal to the sum of a constant and of the Fourier transform of an element of $L_p(-\infty, \infty)$ $[1 \leq p \leq 2]$ if, and only if, $\{g_k(t)\}$ $[k = 1, 2, \cdots]$ is a weakly convergent sequence in $L_p(-\infty, \infty)$.*

Trivially the weak convergence of $\{g_k(t)\}$ in L_1 is a much stronger condition than (9.2) (A), and obviously there is strict equality in the cases concerned in Theorem 4(A) and Theorem 4′ $(p = 1)$, if $F(x)$ is required to be continuous in $(-\infty, \infty)$. The constants are zero if we add the conditions $F(x) \to 0$ as $x \to \infty$ for $p = 1$, $(1 + x)^{-1}F(x) \in L_1(0, \infty)$ for $1 < p \leq 2$.

For completeness we state results which have been proved in another paper.[18]

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[18] H. Kober, *Journal of the London Mathematical Society*, vol. 19 (1944), pp. 144-152. In this paper Theorems 4 and 4′ have been stated without proof. The condition of weak convergence of $\{g_k(t)\}$ in L_1 $(-\infty, \infty)$ can be replaced by the following one: given $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that, uniformly with respect to k $(k = 1, 2, \cdots)$. $|\int_E g_k(t)\,dt| < \epsilon$ whenever the set E satisfies the condition $|\int_E [dt/(1 + t^2)]| < \delta$. There is an analogous result for Fourier series, better than the known ones with respect to sufficient conditions: Let $0 < r < 1$,

$$s_n(x) = \sum_{k=-n}^{n} a_k e^{ikx}, \qquad \sigma_n(x) = [1/(n+1)] \sum_{j=0}^{n} s_j(x), \qquad u(r, x) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{inx}.$$

Then $\Sigma a_n e^{inx}$ is the Fourier series of a function $f(x) \in L_1(-\pi, \pi)$ if, and only if, there

THEOREM 5. *If $\alpha(t)$ is a normalized function of bounded variation over $(-\infty, \infty)$, and if $F(x)$ is represented in the form (8.1), then*

$$(9.3)\quad \tfrac{1}{2}\{\alpha(t+0)-\alpha(t-0)\} = (2\pi)^{-\frac{1}{2}} \lim_{s\to\infty} s \int_{-\infty}^{\infty} \frac{e^{ixt}F(x)}{x^2+s^2}\,dx \qquad [-\infty < t < \infty].$$

$$(9.4)\quad \alpha(t)-\alpha(0+)\operatorname{sgn} t = (2\pi)^{-\frac{1}{2}} \lim_{k\to\infty} \int_0^t du \int_{-\infty}^{\infty} \frac{F(x)\,dx}{(1-iux/k)^{k+2}} \qquad \left[\begin{matrix} -\infty < t < \infty \\ \operatorname{sgn} 0 = 0 \end{matrix}\right].$$

THEOREM 5′. *If, for $1 \le p < \infty$, $F(x)$ is the $\mathfrak{M}^q$ Fourier transform of an element $f(t)$ of $L_p(-\infty, \infty)$ $[1/q + 1/p = 1]$, then*

$$(9.5)\quad F(x) = (2\pi)^{-\frac{1}{2}} \operatorname*{\mathfrak{M}^q}_{k\to\infty} \int_{-\infty}^{\infty} \frac{f(t)\,dt}{(1+itx/k)^{k+1}},$$

$$(9.6)\quad f(t) = (2\pi)^{-\frac{1}{2}} \operatorname*{\mathfrak{M}^p}_{k\to\infty} \int_{-\infty}^{\infty} \frac{F(x)\,dx}{(1-itx/k)^{k+1}} \qquad [k = 1, 2, \cdots, k\to\infty].$$

For $q = \infty$, i.e. $p = 1$, in (9.5) the limit in mean is to be replaced by the ordinary limit, which is uniform with respect to x in any finite interval.

We observe that (9.6) holds for $p = 1$, while, in general, the familiar inversion formula $f(t) = (2\pi)^{-\frac{1}{2}}\mathfrak{M}^p \int e^{itx}F(x)\,dx$ holds for $1 < p < \infty$ only.

From the latter results the necessity of the conditions in Theorems 4(B) and 4′ follows easily, observing that strong convergence in $L_p(-\infty, \infty)$ implies weak convergence and implies therefore also (9.2), and that $g_k[1; t] \equiv 0$ $[k = 1, 2, \cdots]$. Hence we have only to prove the necessity of (9.2) (A), and the sufficiency of the conditions.

10. Proof of theorem 4(A). First we show that the condition is sufficient.

Since $g_k(x)$ is finite for almost all x, we deduce that $(1+t^2)^{-1}F(t)$

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is a sequence $n_1, n_2, \cdots [n_j < n_{j+1}]$, or $r_1, r_2, \cdots [r_j \to 1$ as $j \to \infty]$, such that one of the following alternative conditions is satisfied:

(A) $\{\sigma_{n_j}(x)\}$ is a weakly convergent sequence in $L_1(-\pi, \pi)$.

(B) Given $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that, uniformly with respect to j, $\left|\int_E \sigma_{n_j}(x)\,dx\right| < \epsilon$ whenever $m(E) < \delta$ [E a set in $(-\pi, \pi)$].

(C) $\{u(r_j, x)\}$ is a weakly convergent sequence in $L_1(-\pi, \pi)$.

(D) $\left|\int_E u(r_j, x)\,dx\right| < \epsilon$ whenever $m(E) < \delta = \delta(\epsilon)$.

belongs to $L_1(-\infty, \infty)$, taking $k=1$. Now we form the function $\Phi_2(x)$ defined by (1.42), and put $z=-is$ where $s>0$. Then we have

$$(10.1)\qquad \frac{d^k}{ds^k}\Phi_2(-is) = (-1)^k i^{k+1}\frac{k!}{2\pi}\int_{-\infty}^{\infty}\frac{F(t)\,dt}{(t+is)^{k+1}} \qquad [k=1,2,\cdots],$$

the differentiation under the integral sign being justified by integrating. Let $h(t)$ be defined in $(0, \infty)$, and let

$$(10.2)\qquad L_{k,u}h = \frac{(-1)^k}{k!}\left(\frac{k}{u}\right)^{k+1} h^{(k)}\left(\frac{k}{u}\right) \qquad \left[\begin{matrix} h^{(k)}(t) = \dfrac{d^k h(t)}{dt^k} \\ k=1,2,\cdots \end{matrix}\right]$$

denote the Post-Widder operator. From (10.1) we have

$$\int_0^{\infty} |\, L_{k,u}[\Phi_2(-is)]\,|\,du = \frac{1}{2\pi}\int_0^{\infty} du\,\Big|\int_{-\infty}^{\infty}\frac{F(t)\,dt}{(1-ik^{-1}tu)^{k+1}}\Big|.$$

In consequence of the hypothesis, the right side is bounded with respect to k. This condition, however, together with the existence of derivatives of all orders of $\Phi_2(-is)$ for $0<s<\infty$, is necessary and sufficient [19] for the representation of $\Phi_2(-is)$ in the form

$$\Phi_2(-is) = \int_0^{\infty} e^{-st}\,d\beta(t) \qquad \left(\begin{matrix} s>0, \\ \displaystyle\int_0^{\infty} |\,d\beta(t)| < \infty \end{matrix}\right).$$

Both sides are analytic functions of s for $\Re(s)>0$. Taking $s=\sigma+ix$ and $\sigma\to 0$, the left side tends, for almost all x in $(-\infty, \infty)$, to the limit-function $F_2(x)$ (see Theorem 2), while the right side tends to $\int_0^{\infty} e^{-ixt}\,d\beta(t)$ for all x. Hence

$$F_2(x) \equiv \int_0^{\infty} e^{-ixt}\,d\beta(t)$$

for almost all x. In a similar way we deduce that

$$\phi_1(is) = \int_0^{\infty} e^{-st}\,d\gamma(t); \quad F_1(-x) \equiv \int_0^{\infty} e^{-ixt}\,d\gamma(t), \quad F_1(x) \equiv \int_{-\infty}^{0} e^{-ixt}\,d\tilde{\gamma}(t)$$

$$\left[\begin{matrix} \Re(s)>0; \\ \displaystyle\int_0^{\infty} |\,d\gamma(t)| < \infty, \\ \tilde{\gamma}(t) = -\gamma(-t) \end{matrix}\right].$$

Observing that $F(x) = F_1(x) + F_2(x)$, it is easy to complete the proof.

To prove the necessity of the condition, we compare the resolution of $F(x)$

[19] D. V. Widder, *loc. cit.*, Ch. 7, Theorem 12a.

into two components, based on the representation (8.1), with that given by Theorems 2′, 2″ (a). Thus we have

$$\Phi_2(-is) = (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{-ts} d\alpha(t) + A,$$

$$\Phi_1(is) = (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{-ts} d\{-\alpha(-t)\} - A.$$

Applying the Widder theorem, we deduce (9.2 A), which completes the proof.

11. The case (B) is treated in a similar way. We need some lemmas.

LEMMA 5. *Let $F(t)$ satisfy* (9.2) (A) *or* (B). *Then both $\Phi_1(is)$ and $\Phi_2(-is)$ tend to finite limits as $s \to \infty$.*

LEMMA 6. *Let $1 < p < \infty$, $1/p + 1/q = 1$, and let $H(-x)$ be the $\mathfrak{M}^q$ Fourier transform of a function $h(t) \in L_p(0, \infty)$* [i.e., *$h(t) \equiv 0$ for $t < 0$*]. *Then $H(x)$ is the limit-function of an element $H(z)$ of $\mathfrak{H}_q$, defining $H(z)$ by*

$$(11.1) \qquad (2\pi)^{-\frac{1}{2}} H(is) = \int_0^\infty e^{-st} h(t)\, dt \qquad [\Re s > 0].$$

LEMMA 7.[20] *If* (i) *$f(s)$ has derivatives of all orders for $0 < s < \infty$ and* (ii) *$f(\infty) = 0$ and* (iii), *for some finite $p > 1$,*

$$(11.2) \qquad \int_0^\infty |\, L_{k,u} f\,|^p du < M \qquad [k = 1, 2, \cdots],$$

then $f(s)$ is representable in the form

$$f(s) = \int_0^\infty e^{-st} h(t)\, dt \qquad [0 < s < \infty],$$

where $h(t) \in L_p(0, \infty)$. The conditions are necessary.

Proof of Lemma 5. We need only show that the function

$$\chi(s) = 2\pi \frac{d}{ds} \Phi_1(is) = \int_{-\infty}^\infty \frac{F(t)\, dt}{(t - is)^2}$$

belongs to $L_1(1, \infty)$. Certainly $s^{2-2/p}\chi(s) \in L_p(0, \infty)$ $[1 \leq p \leq 2]$; for

$$\int_0^\infty s^{2p-2} |\, \chi(s)\,|^p ds = \int_{-\infty}^0 du \left| \int_{-\infty}^\infty \frac{F(t)\, dt}{(1 - itu)^2} \right|^p = \int_{-\infty}^\infty |\, g_1(u)\,|^p du < M.$$

Using Hölder's inequality if $p > 1$, we arrive at the required result.

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[20] D. V. Widder, Ch. 7, Theorem 15a.

Lemma 6 follows from known results, due to E. Hille and J. D. Tamarkin,[21] and to G. Doetsch.[22]

Proof of the sufficiency of the condition (*case* B). By Lemma 5, $\Phi_2(-i\infty)$ exists and is finite. The function $\Phi_2(-is) - \Phi_2(-i\infty)$ satisfies the hypotheses of Lemma 7. Hence there is a function $h_2(t) \in L_p(0, \infty)$ such that

$$\Phi_2(-is) = \int_0^\infty e^{-st} h_2(t)\,dt + \Phi_2(-i\infty) \qquad [s > 0].$$

By Theorem 2 and Lemma 6, therefore, we have $F_2(x) = H(-x) + \Phi_2(-i\infty)$, where $H(x)$ is defined by

$$H(-x) = \mathfrak{M}^q \int_0^\infty e^{-ixt} h_2(t)\,dt.$$

Similarly

$$F_1(x) = \mathfrak{M}^q \int_{-\infty}^0 h_1(t) e^{-ixt}\,dt + \Phi_1(i\infty) \qquad [h_1(t) \in L_p(-\infty, 0)].$$

Using Theorem 2, we arrive at the required result.

12. Proof of Theorem 4′. The result for $p > 1$ is now evident. We are left to show that for the case $p = 1$ the condition is sufficient. We need a result which can be considered as a generalisation of a Widder theorem.[23]

LEMMA 8. *If* (i) $f(s)$ *has derivatives of all orders for* $0 < s < \infty$ *and* (ii) $f(\infty) = 0$ *and* (iii) $\{L_{k,u}f\}$ $[k = 1, 2, \cdots]$ *is a weakly convergent sequence in* $L_1(0, \infty)$, *then* $f(s)$ *is representable in the form* $f(s) = \int_0^\infty e^{-st} g(t)\,dt$, *where* $0 < s < \infty$ *and* $g(t) \in L_1(0, \infty)$. *The conditions are necessary.*

In the Widder theorem, $\{L_{k,u}f\}$ is required to be a strongly convergent sequence in $L_1(0, \infty)$. Therefore we need only show that the conditions are sufficient. If $g_0(u)$ is the element of $L_1(0, \infty)$ to which $\{L_{k,u}f\}$ converges weakly, and if $f_0(s) = \int_0^\infty e^{-st} g_0(t)\,dt$, then, by the Widder theorem, the $L_{k,u}f_0$ converge strongly in $L_1(0, \infty)$; in fact they converge to $g_0(t)$. Hence they converge to $g_0(t)$ also weakly; therefore the sequence $\{L_{k,u}[f - f_0]\}$ converges weakly as well, and so

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[21] *Loc. cit.*, Lemma 4.2.

[22] *Mathematische Zeitschrift*, vol. 42 (1937), pp. 263-286.

[23] D. V. Widder, Ch. 7, Theorem 17a.

$$\int_0^\infty |L_{k,u}[f-f_0]|\,du < M \qquad [k=1,2,\cdots].$$

Consequently $f(s)-f_0(s)$ is representable in the form [24]

$$f(s)-f_0(s)=h(s)=\int_0^\infty e^{-st}d\alpha(t) \qquad [s>0,\ \int_0^\infty |\,d\alpha(t)|<\infty].$$

But the $L_{k,u}h$ converge to the null-element, and we have [25]

$$\alpha(t)=\lim_{j\to\infty}\int_0^t L_{n_j,u}h\,du+\beta(t) \qquad \left[\beta(t)=\begin{matrix}-h(\infty)\\0\end{matrix}\ \text{for}\ \begin{matrix}t=0\\t>0\end{matrix}\right],$$

where $\{n_j\}$ is some sequence tending to infinity. Hence $\alpha(t)=\beta(t)$. Since $h(\infty)=f(\infty)-f_0(\infty)=0-0=0$, we have $\alpha(t)=0$ $[0\leqq t<\infty]$ and, therefore, $f(s)-f(s_0)=0$, which proves the lemma.

Using this result, the proof can now be completed by arguments advanced in the preceding section.

13. The Stieltjes transformation

$$Tf=T[f(t);x]=\frac{1}{\pi}\int_0^\infty \frac{f(t)\,dt}{t+x} \qquad [0<x<\infty]. \tag{13.1}$$

A solution of the Stieltjes problem in terms of functions of a real variable has been given by D. V. Widder, also for the case $\bar{T}f=\pi^{-1}\int_0^\infty (t+x)^{-1}df(t)$. Here we shall deal with (13.1) only, using analytic functions, and shall generalise a result known for the case $f(t)\,\epsilon\,L_2(0,\infty)$.[26]

THEOREM 6. (A) *If* (i) *both* $g(z)$ *and* $g(-z)$ *belong to* $\mathfrak{H}_p$ $[1\leq p<\infty]$, *and if* (ii) $g(z)$ *is analytic for* $0<x<\infty$, *then the equation* $g(x)=Tf$ *has a unique solution* $f(t)\,\epsilon\,L_p(0,\infty)$. *The conditions* (ii) *and, for* $1<p<\infty$, (i) *are necessary.*

(B) *If* (i) *both* $(z+i)^{-1}g(z)$ *and* $(z+i)^{-1}g(-z)$ *belong to* $\mathfrak{H}_1$, *and* (ii) $g(z)$ *is analytic for* $0<x<\infty$, *then* $g(x)=Tf$ *has a unique solution* $f(t)$ *such that* $(1+t)^{-1}f(t)$ *is integrable on* $(0,\infty)$.

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[24] D. V. Widder, Ch. 7, Theorem 12a.

[25] D. V. Widder, p. 307.

[26] E. C. Titchmarsh, *loc. cit.*, 11.8. It can be shown that the Titchmarsh condition is equivalent to those of Theorem 6 by the following result: $F(z)$ belongs to $\mathfrak{H}_p$ $[0<p<\infty]$ if and only if $F(z)$ is analytic for $y>0$ and if, uniformly for $0<\theta<\pi$,

$$\int_0^\infty |F(re^{i\theta})|^p\,dr<M^p.$$

In both cases the solution is

$$(13.2)\quad f(t) = \frac{1}{2i} \lim_{y \to 0} \{g(-t-iy) - g(-t+iy)\} \left[\begin{matrix} y > 0, \\ -\infty < t < \infty \end{matrix} \right].$$

Proof of the uniqueness of the solution. It will suffice to show that the conditions $(1+t)^{-1} f_j(t) \in L_1(0, \infty)$ and $Tf_j(t) = g(x)$ $[j = 1, 2]$ imply $f_1(t) \equiv f_2(t)$ $[0 < t < \infty]$. Let $f_1(t) - f_2(t) = f(t)$, $\gamma(z) = T[f(t); z]$. Then $\gamma(z)$ is analytic for $|\arg z| < \pi$, while $\gamma(x) = Tf_1 - Tf_2 = 0$ for $0 < x < \infty$. Hence $\gamma(z)$ vanishes identically. Taking $F(t) = 0$ for $t < 0$, $F(t) = f(t)$ for $t > 0$, and using (1.41) and Theorem 2, we have $F(t) = F_1(t) + F_2(t)$, and $F_j(z) = (-1)^{j-1}(2i)^{-1}\gamma(-z)$ for $y > 0$ $(j = 1)$ or $y < 0$ $(j = 2)$, respectively. Thus $F_j(z)$ vanishes identically; so does, therefore, $F(t)$ and $f_1(t) - f_2(t) = f(t)$.

14. Proof of part (A) of the theorem. The necessity of (ii) is evident. So is that of (i) for $1 < p < \infty$ by (3C), taking $F(t) = f(t)$ for $t > 0$, $= 0$ for $t < 0$. If, for $p = 1$, we require further that $f(t) \in L_1(0, \infty)$ and that $\mathfrak{H}f \in L_1(-\infty, \infty)$, then the necessity of (i) follows from Theorem 2′.

The function $f(t)$, defined by (13.2), has the properties required. Since $g(z)$ is analytic for $0 < x < \infty$, $f(t)$ vanishes for $t < 0$. In consequence of (i), $f(t)$ belongs to $L_p(0, \infty)$. Taking

$$\psi_1(t) = \lim_{y \to 0} g(-t-iy), \qquad \psi_2(t) = \lim_{y \to 0} g(-t+iy) \qquad [y > 0],$$

we have [27]

$$\mathfrak{H}\psi_1 = i\psi_1, \qquad \mathfrak{H}\psi_2 = -i\psi_2$$

as $\psi_1(t)$ and $\psi_2(-t)$ are limit functions of elements of $\mathfrak{H}_p$. Hence

$$f(t) = \frac{1}{2i}(\psi_1(t) - \psi_2(t))$$

$$\mathfrak{H}[f(t); x] = \tfrac{1}{2}(\psi_1(x) + \psi_2(x)) \in L_p(-\infty, \infty).$$

In the latter equation the term on the left is equal to $T[f(t); -x]$ for $x < 0$ and the term on the right to $g(-x)$ which proves the case (A) of the theorem. The case (B) is treated in a similar way, using Lemma 10 (**15**).

15. Generalisation of known results in $\mathfrak{H}_p$ and on Hilbert's operator.

LEMMA 9. *If* $(z+i)^{-1}F(z) \in \mathfrak{H}_1$ *or* $(z+i)^{-2}F(z) \in \mathfrak{H}_1$, *respectively, then*

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[27] H. Kober, *Bulletin of the American Mathematical Society*, vol. 48 (1942), *loc. cit.*, Lemma 4.

$$F(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(t)\,dt}{t-z} \quad or \quad F(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}F(t)\left(\frac{1}{t-z}-\frac{1}{t+i}\right)dt$$

for $y>0$, *while the integrals vanish identically for* $y<0$.

LEMMA 10. *The function* $(t+i)^{-j}F(t)$ $(j=1,2)$ *is the limit-function of an element of* $\mathfrak{H}_1$ *if, and only if,* $(1+|t|)^{-j}F(t)\,\epsilon\,L_1(-\infty,\infty)$ *and* $\mathfrak{H}F=iF$ *or* $\mathfrak{R}F=i(F-a_0)$, *respectively, where* $a_0=\pi^{-1}\int_{-\infty}^{\infty}(1+t^2)^{-1}F(t)dt$.

LEMMA 11. *If* (i) $(1+|t|)^{-j}F(t)$ $(j=1,2)$ *and* (ii) $(1+|x|)^{-1}\mathfrak{H}F$ *or* $(1+x^2)^{-1}\mathfrak{R}F$, *respectively, belong to* $L_1(-\infty,\infty)$, *then*

$$\mathfrak{H}^2F=-F(x)\ [j=1] \quad \text{or} \quad \mathfrak{R}^2F=-F(x)+a_0\ [j=2].$$

If $(z+i)^{-1}F(z)\,\epsilon\,\mathfrak{H}_1$, then the statement concerned in Lemma 9 is deduced from the representation of a function $g(z)\,\epsilon\,\mathfrak{H}_1$ by its proper Cauchy integral.[28] We take $g(z)=(z+i)^{-1}F(z)$ and observe that $g(z)\,\epsilon\,\mathfrak{H}_1$ implies that $\int_{-\infty}^{\infty}g(t)\,dt=0$. The other statement is reduced to the previous case by taking $(z+i)^{-1}F(z)=\tilde{F}(z)$.

The proof of Lemma 10 is left to the reader.

To deduce Lemma 11 for $j=2$, we resolve $F(t)$ into its components according to Theorem 2 and (4.4). Thus we have

$$F(t)=F_1(t)+F_2(t), \tag{15.1}$$

$$\mathfrak{R}F=iF_1(t)-iF_2(t). \tag{15.2}$$

By Theorem 2''' (**4**), $F_j(x)$ $(j=1,2)$ is the limit-function of a function $\Phi_j(z)$ such that $(z+i)^{-2}\Phi_1(z)$ or $(z+i)^{-2}\Phi(-z)$ belongs to $\mathfrak{H}_1$. By Lemma 10, therefore, we have

$$\mathfrak{R}F_j=(-1)^{j-1}i\left(F_j(x)-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{F_j(t)\,dt}{t^2+1}\right).$$

Applying the operator $\mathfrak{R}$ to (15.2), we have

$$\mathfrak{R}^2F=-F_1(x)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{F_1(t)\,dt}{t^2+1}-F_2(x)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{F_2(t)\,dt}{t^2+1}=-F(x)+a_0.$$

In a similar way we deduce the other assertion of the lemma.

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[28] Hille-Tamarkin, *loc. cit.*, Theorem 2.1 (ii).

HADAMARD'S FORMULA AND VARIATION OF DOMAIN-FUNCTIONS.*

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1. Hadamard's differential equation for Green's function.

1. The present paper deals with applications of functional analysis to the theory of the logarithmic potential and analytic functions. We consider domains D in the complex x-plane bounded by a finite number of proper continua $C_\nu (\nu = 1, \cdots, n)$. If x and y are points in D, Green's function $g(x;y)$ of D is defined in the following way:

a. $g(x;y)$ is a harmonic function of x throughout D, the point $x = y$ excepted; there however $g(x;y) + \log|x-y|$ is harmonic.

b. If x converges to a boundary continuum C_ν, $g(x;y)$ converges to zero.

The existence of $g(x;y)$ is assured for every domain D; $g(x;y)$ is harmonic in y also and satisfies the symmetry condition $g(x;y) = g(y;x)$. Its importance for Dirichlet's problem is well known. If D is simply connected, $g(x;y)$ is related also to the problem of mapping D onto the exterior E of the unit circle. For, let $p(x;y)$ be an analytic function of x in D, such that $g(x;y) = R\{p(x;y)\}$; then $\phi(x;y) = \exp\{p(x;y)\}$ maps D conformally on E so that the point y goes into infinity.

Using the theory of orthogonal functions we may determine the Green's function in the form of an infinite series for any domain whose boundary satisfies certain general conditions. (See Bergman [1] 1, 2.) On the other hand consideration of the rôle of Green's formula in the theory of functions and potential theory indicates the desirability of obtaining various representations of it and in particular those which show the dependence of this formula on the domain in which it is defined.

There are however only a few elementary domains D with an explicitly known Green's function. Nevertheless it is possible to calculate Green's function (approximately) for domains which are sufficiently near such elementary domains. This is done by means of Hadamard's well known variation formula (see Hadamard 2) which is applicable to every domain D bounded by analytic

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[1] See Bibliography, p. 448.

curves C_ν. Let every point of the boundary $C = \sum_{\nu=1}^{n} C_\nu$ be defined by a parameter s which measures the lengths of the curves successively and runs, therefore, from 0 to l (l = sum of the lengths of all C_ν). Let $\delta n(s) = \epsilon\nu(s)$ be a continuous function of s which determines the normal displacement of each boundary point $z(s)$ of the original domain D. $\delta n(s)$ is taken as positive if the displacement is in the direction of the outer normal with respect to D. In this way, we define a new domain D^* with a new Green's function $g^*(x;y)$. According to Hadamard, we have

$$(1) \qquad g^*(x;y) = g(x;y) + \frac{\epsilon}{2\pi}\int_C \frac{\partial g(z;x)}{\partial n_z}\,\frac{\partial g(z;y)}{\partial n_z}\,\nu(s_z)\,ds_z + o(\epsilon).$$

Here, $\frac{\partial}{\partial n_z}[g(z;x)]$ denotes the derivative of $g(z;x)$ in the direction of the outward normal and $o(\epsilon)$, as usual, a term satisfying the condition $\lim_{\epsilon=0}\frac{1}{\epsilon}o(\epsilon) = 0$. Using the notations of functional calculus (Volterra 1), we may give to (1) the form

$$(1') \qquad \delta g(x;y) = \frac{1}{2\pi}\int_C \frac{\partial g(z;x)}{\partial n}\,\frac{\partial g(z;y)}{\partial n}\,\delta n\,ds.$$

Many properties of $g(x;y)$ and the univalent mapping functions connected with it may be derived from (1′) (see Lévy, Julia, Biernacki). This formula, however, loses its meaning if the initial domain D is not bounded by smooth curves; on the other hand, in this case a well defined Green's function also exists. Thus (1′) is not applicable to extremum problems concerning Green's function; for one can not be sure that the domain D, belonging to the extremal function, satisfies the suppositions of Hadamard's formula.

2. We shall now transform formula (1′) into such a form that it may be applied to the most general domain D. We use the artifice of specializing the variation δn and, by means of partial integration, expressing $\delta g(x;y)$ by values of $g(x;y)$ and its derivatives from the *interior* of D. For this purpose, we choose a fixed point z_0 in D and consider the representation

$$(2) \qquad x^* = x + \frac{e^{2i\phi}\rho^2}{x - z_0}, \quad \rho > 0, \qquad 0 \leq \phi < 2\pi$$

of the x-plane. It transforms the circumference $|x - z_0| = \rho$ into the segment $< -2\rho e^{i\phi}, +2\rho e^{i\phi} >$ and is univalent in its exterior $|x - z_0| > \rho$. For ρ sufficiently small, this representation is univalent on all curves C_ν and transforms them in a one-to-one manner into neighboring curves C^*_ν which

enclose a new domain D^* of the x-plane. Let $g^*(x;y)$ denote the Green's function of D^*; we shall compute it from $g(x;y)$ by aid of (1').

We remark that for $z \subset C_\nu$, $y \subset D$

$$g(z^*;y) = \frac{\partial g(z;y)}{\partial n}\,\delta n + o(\rho^2); \tag{3}$$

here δn is the normal shift of C_ν at the point z, caused by the variation (2). δn is of order ρ^2. Hence (1) has the form

$$g^*(x;y) = g(x;y) + \frac{1}{2\pi}\int_C \frac{\partial g(z;x)}{\partial n}\, g(z^*(z);y)\,ds + o(\rho^2). \tag{4}$$

Let y_1 be the point which is transformed by (2) into y; since $g(z;y_1)$ vanishes on the boundary of D, we may write instead of (4)

$$\delta g(x;y) = \frac{1}{2\pi}\int_C \frac{\partial g(z;x)}{\partial n}\,[g(z^*;y) - g(z;y_1)]\,ds + o(\rho^2): \tag{4'}$$

Now, $g(z^*;y) - g(z;y_1)$ is harmonic in the domain D_0, obtained from D by discarding the interior of the circumference $K_\rho \equiv (|\,x - z_0\,| = \rho)$. Hence, Green's formula may be appplied to D_0, yielding

$$\begin{aligned}\delta g(x;y) = \frac{1}{2\pi}\int_{K\rho} \Big\{ &\frac{\partial g(z;x)}{\partial n}[g(z^*;y) - g(z;y_1)] \\ &- g(z;x)\,\frac{\partial}{\partial n}\,[g(z^*;y) - g(z;y_1)] \Big\}\, ds + g(x;y_1) - g(x^*;y) + o(\rho^2)\end{aligned} \tag{5}$$

where the normal derivative is to be taken in the direction of increasing ρ. We suppose x, y, y_1 to lie in the exterior of K_ρ and get, therefore, from Green's formula

$$\frac{1}{2\pi}\int_{K\rho} \left\{ \frac{\partial g(z;x)}{\partial n}\, g(z;y_1) - g(z;x)\,\frac{\partial g(z;y_1)}{\partial n} \right\} ds = 0. \tag{5'}$$

Thus, there remains

$$\begin{aligned}\delta g(x;y) = \frac{1}{2\pi}\int_{K\rho} \left\{ \frac{\partial g(z;x)}{\partial n}\, g(z^*;y) - g(z;x)\,\frac{\partial g(z^*;y)}{\partial n} \right\} ds \\ + g(x;y_1) - g(x^*;y) + o(\rho^2)\end{aligned} \tag{6}$$

which expresses the variation of $g(x;y)$ by means of the values of $g(x;y)$ on the circumference K_ρ, interior to D, only. This formula may be simplified by means of a development in series. We introduce an auxiliary function $p(x;y)$ analytic in x and satisfying

$$g(x;y) = R\{p(x;y)\}. \tag{7}$$

$p(x;y)$ is not necessarily uniform in D and contains an arbitrary additive imaginary constant depending on y. On K_ρ we have $z = z_0 + \rho e^{i\tau}$ so that, on applying Taylor's theorem to $p(x;y)$ and by (7)

$$(8) \qquad g(z^*;y) = g(z_0;y) + R\{\rho(e^{i\tau} + e^{i(2\phi-\tau)})p'(z_0;y)\} + O(\rho^2),$$

$$(8') \qquad \frac{\partial}{\partial n} g(z^*;y) = R\{(e^{i\tau} - e^{i(2\phi-\tau)})p'(z_0;y)\} + O(\rho),$$

$$(8'') \qquad g(z;x) = g(z_0;x) + R\{\rho(e^{i\tau}p'(z_0;x)\} + O(\rho^2),$$

$$(8''') \qquad \frac{\partial}{\partial n} g(z;x) = R\{e^{i\tau}p'(z_0;x)\} + O(\rho),$$

$$(8^{IV}) \qquad g(x;y_1) = g(x;y) - R\left\{\frac{e^{2i\phi}\rho^2}{y - z_0} p'(y;x)\right\} + o(\rho^2),$$

$$(8^{V}) \qquad g(x^*;y) = g(x;y) + R\left\{\frac{e^{2i\phi}\rho^2}{x - z_0} p'(x;y)\right\} + o(\rho^2).$$

The dash denotes differentiation of $p(x;y)$ with respect to its first argument and $O(\epsilon)$ a term satisfying the condition that $(1/\epsilon)O(\epsilon)$ remains bounded if $\epsilon \to 0$.

Introducing formulae (8) into (6) we obtain

$$(9) \qquad \delta g(x;y) = -R\left\{e^{2i\phi}\rho^2\left[\frac{p'(x;y)}{x - z_0} + \frac{p'(y;x)}{y - z_0}\right]\right\} + \frac{1}{2\pi}\int_0^{2\pi} 2R\{e^{i\tau}p'(z_0;x)\} \cdot R\{e^{i(2\phi-\tau)}p'(z_0;y)\}\rho^2 d\tau + o(\rho^2)$$

and by an easy transformation

$$(10) \qquad g^*(x;y) = g(x;y) + R\left\{e^{2i\phi}\rho^2\left[p'(z_0;x)p'(z_0;y) - \frac{p'(x;y)}{x - z_0} - \frac{p'(y;x)}{y - z_0}\right]\right\} + o(\rho^2).$$

In virtue of the identity

$$(10') \qquad g^*(x^*;y^*) = g(x;y) + R\left\{e^{2i\phi}\rho^2\left[\frac{p'(x;y)}{x - z_0} + \frac{p'(y;x)}{y - z_0}\right]\right\} + o(\rho^2),$$

we may put (10) into the simple form (Schiffer 3)

$$(11) \qquad g^*(x^*;y^*) = g(x;y) + R\{e^{2i\phi}\rho^2 p'(z_0;x)p'(z_0;y)\} + o(\rho^2)$$

giving the law of variation of $g(x;y)$ under the particular transformation (2) of D.

(11) was derived from (1) by partial integration. The latter formula is only valid for analytically bounded domains; the same holds, therefore, for (11). But the most general domain D may be approximated arbitrarily

by analytically bounded domains; Green's function of the approximating domain with all its derivatives tends to the corresponding terms of the given domain, uniformly in every interior part. $o(\rho^2)$ depends only upon $g(x;y)$ and its derivatives at interior points of D; hence it can be estimated uniformly. Thus the validity of (11) can be derived for the most general domains D. This fact proves (11) to be superior to (1′), and numerous applications of this formula are possible in extremum problems, concerning the theory of the logarithmic potential and conformal representation, as will be seen below.

Another application of our method is as follows: In the paper (Schiffer 4) the relation between Green's function and the kernal function (see Bergman 1, §VII) is derived. The methods used in the present paper can be employed in investigating the kernal function and lead to new results concerning the behavior of this function in simply- and multiply-connected domains.

2. Further solutions of the variational equation (11).

1. The differential equation (1′) for Green's function admits as solution also other domain functions depending on two variables x, y (see Lévy). If the values of such a function are known for an elementary initial domain, say a circle, it can be calculated in principle for *every* other domain by means of (1′). But it will, in general, be difficult to characterize such a function geometrically, independently of the definition by means of a variational equation. On the other hand, we shall define now an important family of domain functions which occur in the theory of conformal representation of multiply connected domains and which satisfy (11).

In the theory of Green's function the boundary condition plays a decisive part. We shall define a more general type of boundary condition which renders the same service and will be called henceforth the type N. It is characterized by the following property:

Let C_ν $(\nu = 1, \cdots, n)$ be a system of smooth curves enclosing a domain D and let $z^*(z)$ be a conformal representation in the neighborhood of all the C_ν, transforming them into a new system of curves C^*_ν which encloses the domain D^*. A boundary condition is said to be of type N, if it assures in this case the following two facts: $\phi(x)$ satisfying this condition with respect to D, $\psi^*(x)$ with respect to D^*, implies that

a. $\phi(x)$ has continuous derivatives on C_ν, and $\psi^*(x)$ on C^*_ν.

b. The following relation is always valid:

$$\int_C \{\phi(z) \frac{\partial}{\partial n_z} \psi^*(z^*(z)) - \psi^*(z^*(z)) \frac{\partial}{\partial n_z} \phi(z)\} ds_z = 0. \tag{12}$$

Next, domain functions $\gamma(x;y)$ of D with the following properties will be considered:

a. $\gamma(x;y)$ is, for $y \subset D$ fixed, a harmonic function of $x \subset D$, the point $x = y$ excepted.

b. In the neighborhood of $x = y$, the expression $\gamma(x;y) + \log|x-y|$ is bounded.

c. The function $\gamma(x;y)$ depends continuously on D, uniformly with respect to x.

d. $\gamma(x;y)$ satisfies as function of x a boundary condition of type N.

We assert that on these conditions $\gamma(x;y)$ has a variation formula (11).

First, it is obvious that $\gamma(x;y)$ is symmetric with respect to both its arguments. For, on one hand, Green's identity yields in the case of a smooth boundary C

$$(13)\qquad \frac{1}{2\pi}\int_C \{\gamma(z;x)\frac{\partial}{\partial n}\gamma(z;y) - \gamma(z;y)\frac{\partial}{\partial n}\gamma(z;x)\}ds = \gamma(x;y) - \gamma(y;x),$$

while, on the other hand, the boundary condition of $\gamma(x;y)$ ensures, by virtue of (12), that this integral vanishes; we have only to put $\phi(z) = \gamma(z;x)$, $z^*(z) = z$, $\psi^*(z) = \gamma(z;y)$. Hence, we have proved the symmetry of $\gamma(x;y)$ for domains D, bounded by smooth curves. But in view of the continuity of $\gamma(x;y)$ in dependence on the domain, this establishes the same property for domains with general boundary.

Let D denote a domain with smooth boundary curves C_ν; by means of the variation (2) it is transformed into D^* with the corresponding domain function $\gamma^*(x;y)$. As in **1**, 2, we construct the domain D_0 by discarding from D the interior of the circumference $K_\rho \equiv (|x - z_0| = \rho)$. In D_0, the function $d(x;y) = \gamma^*(x^*(x);y^*(y)) - \gamma(x;y)$ is harmonic in both its arguments, the logarithmic pole $x = y$ being cancelled by subtraction. Hence, Green's identity yields easily

$$(14)\qquad \frac{1}{2\pi}\int_{C+K_\rho} \{\gamma(z;x)\frac{\partial}{\partial n}d(z;y) - d(z;y)\frac{\partial}{\partial n}\gamma(z;x)\}ds = d(x;y) = \gamma^*(x^*;y^*) - \gamma(x;y).$$

But γ and γ^* satisfy a boundary condition of type N which shows that the integral over C vanishes. Combining this fact with Green's identity

$$(15)\qquad \frac{1}{2\pi}\int_{K_\rho} \{\gamma(z;x)\frac{\partial}{\partial n}\gamma(z;y) - \gamma(z;y)\frac{\partial}{\partial n}\gamma(z;x)\}ds = 0$$

which holds for x and y in D_0, we get finally from (14)

$$(15') \qquad \gamma^*(x^*;y^*) = \gamma(x;y) + \frac{1}{2\pi}\int_{K_\rho}\{\gamma(z;x)\frac{\partial}{\partial n_z}\gamma^*(z^*;y^*) - \gamma^*(z^*;y^*)\frac{\partial}{\partial n_z}\gamma(z;x)\}ds_z$$

where the normal derivative is to be taken in the outward direction with respect to D_0. If we want to give to n the direction of increasing radius ρ, as is usual, we get

$$(15'') \qquad \gamma^*(x^*;y^*) = \gamma(x;y) + \frac{1}{2\pi}\int_{K_\rho}\{\gamma^*(z^*;y^*)\frac{\partial}{\partial n_z}\gamma(z;x) - \gamma(z;x)\frac{\partial}{\partial n_z}\gamma^*(z^*;y^*)\}ds_z.$$

Thus, we have expressed the variation of $\gamma(x;y)$ by an integral taken along a circumference interior to D as we did in (6) with respect to $g(x;y)$. We may perform now the same formal transformations and developments into series as in **1**, 2; these hold for $\gamma(x;y)$ too, since we have used there only the symmetry and harmonicity of $g(x;y)$. We introduce a function $\phi(x;y)$, analytic for given $y \subset D$ with respect to its first argument $x \subset D$ and satisfying

$$(16) \qquad \gamma(x;y) = R\{\phi(x;y)\}.$$

In general, $\phi(x;y)$ will not be uniform in D; it possesses additive (imaginary) moduli with respect to circuits around the C_ν. We may perform on it the same operations as we did above on $p(x;y)$ and we get, in complete analogy to (11),

$$(17) \qquad \gamma^*(x^*;y^*) = \gamma(x;y) + R\{e^{2i\phi}\rho^2\phi'(z_0;x)\phi'(z_0;y)\} + o(\rho^2).$$

This is the law of variation of $\gamma(x;y)$ under the transformation (2) of a domain D with a smooth boundary. But in view of the uniform continuity of $\gamma(x;y)$ this formula remains valid for domains with general boundary.

2. Our next task is to point out definite examples of domain-functions of the above type. It is well known that every domain D can be mapped in an infinity of ways on the exterior of a circle, cut along concentric circular slits (Koebe 1, Grötzsch 1). This canonical representation is in many respects a natural generalization of the representation of a simply connected domain on the exterior of a circle.

If $f(x;y)$ is a univalent function of x in a simply-connected domain D and maps D on the exterior E of the unit circle such that $y \subset D$ corresponds to infinity, then the Green's function of D is given by

$$(18) \qquad g(x;y) = \log|f(x;y)|.$$

In the case of a multiply connected domain, however, there exists for every $m = 1, 2, \cdots, n$ a function $f_m(x; y)$, mapping D on the exterior of the unit circle, slit along $n-1$ concentric circular arcs, such that C_m corresponds to the unit circle and y to infinity. Thus there exist n domain functions

$$\gamma_m(x; y) = \log |f_m(x; y)| \tag{19}$$

which are a generalization of Green's function for a simply connected domain. We want to show that each $\gamma_m(x; y)$ has a variation formula (17), in analogy to $g(x; y)$. To prove this, it suffices, in view of 1, to show that the boundary conditions for $\gamma_m(x; y)$ are of type N.

Now, there are two characteristic properties of $\gamma_m(x; y)$: (a) $\gamma_m(z; y)$ is zero for z on C_m and constant on each C_ν. (b) $f_m(x; y)$ being uniform in D, the conjugate function of $\gamma_m(x; y)$, i. e., $I\{\log f_m(x; y)\}$, does not change when x describes a circuit around C_ν $(\nu \neq m)$ which means, in the case of a smooth C_ν,

$$\int_{C_\nu} \frac{\partial}{\partial n_z} \gamma_m(z; y)\, ds_z = 0 \qquad (\nu \neq m). \tag{20}$$

These two facts represent just a boundary condition of type N. For, if any $\phi(z)$ possesses the properties (a) and (b) on the system of smooth curves C_ν and any $\psi^*(z)$ on the corresponding system C^*_ν, then we have for each curve C_ν

$$\int_{C_\nu} \{\phi(z) \frac{\partial}{\partial n_z} \psi^*(z^*(z)) - \psi^*(z^*(z)) \frac{\partial}{\partial n_z} \phi(z)\} ds_z = 0. \tag{12'}$$

In fact, $\phi(z)$ and $\psi^*(z^*(z))$ vanish on C_m; on each other C_ν both functions are constant and may be taken out of integration. Thus, there remain periods of type (20) which vanish. Hence, the boundary conditions considered are of type N and $\gamma_m(x; y)$ satisfies, therefore, a variational equation (17).

3. We mention now another type of canonical representation which may again be considered as a generalization of the representation onto the exterior E of the unit circle in the case of a simply connected domain. It is the representation of D by means of a function $h_m(x; y)$ onto E, slit along $n-1$ radial segments, such that C_m corresponds to the circle and y to infinity (Grötzsch 1, Rengel 1, Koebe 1). Consider now the domain function

$$\varkappa_m(x; y) = \log |h_m(x; y)|. \tag{21}$$

This function too possesses a variation formula (17), since it has boundary conditions of type N. To show this, we remark the following two facts:

(a) $\kappa_m(x;y)$ vanishes on C_m. (b) On every C_ν $(\nu \neq m)$ the conjugate function of $\kappa_m(x;y)$, i. e., $I\{\log h_m(x;y)\}$ is constant. Hence, using Cauchy-Riemann's differential equations, we have for $z \subset C_\nu$ $(\nu \neq m)$

$$\frac{\partial}{\partial n_z}\kappa_m(z;y) = -\frac{\partial}{\partial s_z} I\{\log h_m(z;y)\} = 0. \tag{22}$$

These conditions ensure the validity of relations of type (12) for every smooth C_ν; for, $\kappa^*_m(x^*;y^*)$, too, has a constant conjugate function if z remains on C_ν $(\nu \neq m)$. The boundary conditions (a) and (b) are, therefore, of type N and $\kappa_m(x;y)$ satisfies a variational equation (17).

We may consider other canonical representations of D, by means of univalent functions $l_m(x;y)$ which map D onto E slit by concentric circular arcs and radial segments, such that $y \subset D$ corresponds to infinity, C_m to the circle and the other C_ν to either of the slits. It follows easily that the domain function

$$\lambda_m(x;y) = \log|l_m(x;y)| \tag{23}$$

is of the type $\gamma(x;y)$ characterized in 1. Hence it varies too according to (17).

Thus, we have found various domain functions with the same law of variation as Green's function. The function $p(x;y)$ in (11) can only with difficulty be interpreted geometrically, whereas the functions $\phi(x;y)$, used in (17) for the variation of our domain functions, are the logarithms of univalent functions yielding canonical representations. The next paragraphs will show the advantages which evolve from this fact.

4. Next, we transform (17) in such a way that its relation to Hadamard's formula is exposed. From (17) we obtain easily in virtue of (2)

$$\gamma^*(x;y) = \gamma(x;y) + R\left\{ e^{2i\phi}\rho^2\left[\phi'(z_0;x)\phi'(z_0;y) - \frac{\phi'(x;y)}{x-z_0} - \frac{\phi'(y;x)}{y-z_0}\right]\right\} + o(\rho^2). \tag{17'}$$

Let now C' be a system of smooth curves forming the boundary of an arbitrary partial domain of D which contains x, y and z_0; then the residue theorem yields

$$\frac{1}{2\pi i}\int_{C'} \phi'(z;x)\phi'(z;y)\frac{ds}{z-z_0} = \phi'(z_0;x)\phi'(z_0;y) - \frac{\phi'(x;y)}{x-z_0} - \frac{\phi'(y;x)}{y-z_0}, \tag{24}$$

whence, in view of (17'),

$$(25)\qquad \gamma^*(x;y)=\gamma(x;y)+R\left\{e^{2i\phi}\rho^2\cdot\frac{1}{2\pi i}\int_{C'}\phi'(z;x)\phi'(z;y)\,\frac{ds}{z-z_0}\right\}+o(\rho^2).$$

If the boundary C of D is smooth, we may choose, in particular, $C'=C$; in this case we represent C as in 1, 1 by means of the length parameter s which insures $|z'(s)|=1$ on every C_ν.

We apply (25) to the functions $\gamma_m(x;y)$ defined by (19); we put

$$(19')\qquad \phi_m(x;y)=\log f_m(x;y),\ \text{i. e.,}\ \gamma_m(x;y)=R\{\phi_m(x;y)\}.$$

By definition of $f_m(x;y)$, $\phi_m(x;y)$ has a constant real part for $z\subset C_\nu$. Hence, along every curve C_ν,

$$\frac{d}{ds}\phi_m(z(s);y)=\phi'_m(z;y)\cdot z'$$

is imaginary and its value is, according to Cauchy-Riemann's differential equation,

$$\phi'_m(z;y)z'=-i\frac{\partial}{\partial n_z}\gamma_m(z;y).$$

Hence, we may write (25) in the form

$$(25')\qquad \gamma^*_m(x;y)=\gamma_m(x;y)+\frac{1}{2\pi}\int_C\frac{\partial}{\partial n_z}\gamma_m(z;x)\,\frac{\partial}{\partial n_z}\gamma_m(z;y)R\left\{\frac{i}{z'}\,\frac{e^{2i\phi}\rho^2}{z-z_0}\right\}ds+o(\rho^2).$$

On the other hand, we get obviously for the value of the normal shift by variation (2)

$$(26)\qquad \delta n=R\left\{\frac{i}{z'}\,\frac{e^{2i\phi}\rho^2}{z-z_0}\right\}.$$

Hence, (25′) becomes

$$(27)\qquad \delta\gamma_m(x;y)=\frac{1}{2\pi}\int_C\frac{\partial}{\partial n_z}\gamma_m(z;x)\cdot\frac{\partial}{\partial n_z}\gamma_m(z;y)\delta n\,ds,$$

which is just Hadamard's formula for our domain functions. It is derived from (17) for all variations obtainable by superposition of elementary variations (2).

Let $\psi_m(x;y)$ be an analytic function of x, connected with the functions (21) by $\kappa_m(x;y)=R\{\psi_m(x;y)\}$; for $z\subset C_\nu(\nu\neq m)$, $\psi_m(z;y)$ has a constant imaginary part. Hence, on all these curves

$$(28)\qquad \frac{d}{ds}\psi_m(z(s);y)=\psi'_m(z;y)z'=\frac{\partial}{\partial s_z}\kappa_m(z;y).$$

A transformation of (25), analogous to the above, yields by means of (28)

$$(28') \qquad \delta\kappa_m(x;y) = \frac{1}{2\pi}\int_{C_m} \frac{\partial}{\partial n_z}\kappa_m(z;x)\,\frac{\partial}{\partial n_z}\kappa_m(z;y)\delta n\,ds$$
$$-\frac{1}{2\pi}\int_{C-C_m} \frac{\partial}{\partial s_z}\kappa_m(z;x)\,\frac{\partial}{\partial s_z}\kappa_m(z;y)\delta n\,ds.$$

Comparison of (27) and (28′) shows that the same formula (17) appears in its integral representation in very different forms according to the particular boundary conditions satisfied by $\gamma(x;y)$. This shows the advantage of the unifying formula (17).

3. Applications of the variation formula to Green's function.

1. Green's function $g(x;y)$ may be considered as a measure of D with respect to the pair of points x, y in D. In fact, Hadamard's formula (1′) shows the monotonic behavior of $g(x;y)$ as a domain function; for if δn is always positive, i. e., if D is enlarged, $g(x;y)$ is seen to increase, since along the entire boundary we have $\frac{\partial}{\partial n_z} g(z;x) < 0$.

In the neighborhood of y, we have the development

$$(29) \qquad g(x;y) = \log\frac{1}{|x-y|} + \log\frac{1}{d(y)} + O(|x-y|)$$

where $O(\epsilon) \to 0$ with $\epsilon \to 0$. If $\infty \subset D$, we have further

$$(29') \qquad g(x;\infty) = \log|x| + \log\frac{1}{d(\infty)} + O\left(\frac{1}{|x|}\right).$$

The functional $d(\infty)$ has been used frequently in the theory of conformal representation as a measure for D or its boundary C. $d(\infty)$ is called transfinite diameter, Robin's constant or capacity constant of C, or of the domain D whose boundary is C (see Fekete, Szegö, Nevanlinna). All domains which may be mapped upon each other by univalent functions $f(x)$, normalized as

$$(30) \qquad f(x) = x + a_0 + \frac{a_1}{x} + \cdots$$

at infinity, have the same measure $d(\infty)$. Analogously, $d(y)$ is a conformal invariant with respect to representations of D which map y on itself and have there the derivative 1. If $f(x)$ has at infinity (or at y) the derivative a, however, D is mapped on a domain with the measure $ad(\infty)$ (or $ad(y)$). In particular, all $d(y)$ behave like lengths with regard to homotheties.

If D is simply connected, it is well known that $d(y)$ is the radius of the circle on the exterior of which D may be mapped by a function with the normalization

$$(30') \qquad f(x) = \frac{1}{x-y} + a_0 + a_1(x-y) + \cdots$$

(or (30) if $y = \infty$).

Introducing into (1′) and (11) the developments (29) and (29′) and letting $x \to y$ we get, by an easy comparison of both sides, the general formulas for the variation of $d(y)$:

$$(31) \qquad \delta \log d(y) = -\frac{1}{2\pi}\int_C \left(\frac{\partial g(z;y)}{\partial n_z}\right)^2 \delta n\, ds$$

$$(31') \qquad \log d^*(y^*) = \log d(y) - R\left\{\epsilon^{2i\phi}\rho^2\left[\gamma'(z_0;y)^2 - \frac{1}{(y-z_0)^2}\right]\right\} + o(\rho^2).$$

Both formulas remain valid for $y = \infty$.

(31) shows that $d(y)$ is a functional which decreases as the domain increases. If we consider the domain function

$$(32) \qquad \Gamma(x;y) = 2g(x;y) + \log d(x) + \log d(y)$$

we find, by (1′) and (31), its variation formula

$$(32') \qquad \delta\Gamma(x;y) = -\frac{1}{2\pi}\int_C \left[\frac{\partial g(z;x)}{\partial n} - \frac{\partial g(z;y)}{\partial n}\right]^2 \delta n\, ds$$

which proves that $\Gamma(x;y)$ is a decreasing domain function.

2. To demonstrate the applicability of (31′), we deal now with an extremum problem which will become useful later. We choose a boundary continuum C_m of D and a point $y \subset D$; every conformal representation of D of type (30′) transforms C_m into a new continuum $\tilde{C}_m$ with a certain transfinite diameter $\delta = \delta(\infty)$. We ask for the extremal values which δ can attain, if all functions (30′), univalent in D, are considered.

To solve this problem of distortion, we start with a domain D with boundary curves $\tilde{C}_\nu$, obtained from D by such a univalent function (30′) that δ is maximal. This domain is by no means fixed uniquely; for a representation (30), univalent in the exterior of $\tilde{C}_m$, will not change δ and its superposition on any mapping function (30′) will preserve the normalization (30′). This arbitrariness may be avoided by supposing $\tilde{C}_m$ to be a circle cen-

tered at the origin. The radius of the circle is necessarily δ, since the transfinite diameter of a circle is equal to its radius. We proceed to determine the shape of the remaining boundary continua $\tilde{C}_\nu$ $(\nu \neq m)$. For this purpose, we choose a continuum $\tilde{C}_l$ $(l \neq m)$, a fixed point z_0 on it and a subcontinuum Γ of $\tilde{C}_l$, containing z_0, of transfinite diameter ρ. All functions (30) which are univalent in the exterior of Γ permit a development

$$x^+ = \phi(x) = x + k + \frac{a\rho^2}{x - z_0} + \frac{b\rho^3}{(x - z_0)^2} + \cdots \tag{33}$$

valid in every given domain interior to $\tilde{D}$ for Γ sufficiently small, such that $a, b, \cdots$ have bounds independent of Γ. The superposition of a representation (33) on the original mapping of D on $\tilde{D}$ gives a univalent transformation (30′) of D on a new domain D^+. This has a boundary continuum $C_m{}^+$ arising from $\tilde{C}_m$ by means of (33). Its transfinite diameter $\delta^+ = \delta^+(\infty)$ satisfies, in view of the maximal property of δ, the inequality

$$\delta^+ \leq \delta. \tag{34}$$

If, on the other hand, $\pi(x; y)$ is an analytic function of x such that $R\{\pi(x; y)\}$ is Green's function for the exterior of $\tilde{C}_m$, we may find δ^+ with the aid of (31′). In fact, the function (33) differs on $\tilde{C}_m$ from

$$x^* = x + k + \frac{a\rho^2}{x - z_0} \tag{2′}$$

only in terms of order $o(\rho^2)$. Hence, in view of (31), applicable in the case $C = \tilde{C}_m$, they cause a variation of $\log \delta$ of this order only, in addition to that caused by (2′) and given by (31′). So, we get

$$\log \delta^+ = \log \delta - R\{a\rho^2 \pi'(z_0; \infty)^2\} + o(\rho^2). \tag{35}$$

Since $\tilde{C}_m$ is a circle of radius δ and has as its Green's function $g(x; \infty) = \log \frac{|x|}{\delta}$, we have

$$\pi(x; \infty) = \log x - \log \delta \tag{36}$$

and, finally, the following formula for the variation of $\log \delta$:

$$\log \delta^+ = \log \delta - R\left\{ a\rho^2 \cdot \frac{1}{z_0{}^2} \right\} + o(\rho^2). \tag{35′}$$

Comparing (34) with (35′) we get the inequality

$$R\left\{ a\rho^2 \cdot \frac{1}{z_0^2} \right\} + o(\rho^2) \geq 0, \tag{34'}$$

whatever z_0, Γ and the function (33) may have been.

Now, we have to apply the following lemma (Schiffer 1):

LEMMA. *Let C be a continuum in the x-plane; suppose that there exists an analytic function $s(x) \not\equiv 0$ such that for an arbitrary function (33) univalent in the exterior of an arbitrary subcontinuum Γ of transfinite diameter ρ and containing the arbitrary point z_0 we have*

$$R\{a\rho^2 s(z_0)\} + o(\rho^2) \geq 0. \tag{37}$$

Then C is an analytic curve, expressed by means of a real parameter in the form $x = x(t)$ such that

$$x'(t)^2 s[x(t)] + 1 = 0. \tag{37'}$$

Applying this lemma to (34') we find $\tilde{C}_l$ to be an analytic curve with the differential equation

$$x'(t)^2 x(t)^{-2} + 1 = 0 \tag{37''}$$

whence

$$x(t) = k_l e^{it} \qquad k_l = \text{constant}. \tag{37'''}$$

Thus we have proved that δ attains its maximum for the representation of D which transforms C_m into a circle and the other $n-1$ continua C_ν into concentric circular arcs.

As the existence of extremal functions is insured in the case of the problem considered, this theorem proves anew the possibility of this particular type of canonical conformal representation.

Had we raised the question of the minimum of the transfinite diameter δ of all possible $\tilde{C}_m$, obtained by representations (30'), our method would have been exactly the same. Only, instead of (34), we would have used the inverse inequality leading to

$$R\left\{ a\rho^2 \frac{1}{z_0^2} \right\} + o(\rho^2) \leq 0. \tag{34''}$$

A new application of our lemma shows that in this case also every $\tilde{C}_l$ is an analytic curve satisfying now the differential equation

$$x'(t)^2 x(t)^{-2} = 1 \tag{37 IV}$$

which yields

$$x(t) = \kappa_l e^t \qquad \kappa_l = \text{constant}. \tag{37 V}$$

Hence: The minimum of δ is obtained for the representation of D which transforms C_m into a circle and the remaining C_ν into radial segments.

The extremum problem considered above was so easily solved because the function $\pi(x;\infty)$, appearing in the variation formula (35), is the logarithm of a univalent function. Thus, we could apply an auxiliary mapping and give to $\pi(x;\infty)$ a suitable particular form. Most variational problems, concerning the transfinite diameter of a multiply connected domain, are considerably more difficult because the function $p(x;y)$, occurring in (31′), is not connected with univalent functions. In the variational rule of $\gamma_m(x;y)$ and $\kappa_m(x;y)$, however, there appears always the logarithm of a univalent function, which facilitates obviously the treatment of these domain functions.

3. The method of treating extremum problems, just applied, permits an important numerical application in the theory of mapping simply connected domains. Consider all functions (30) which are univalent in $|x|>1$. They map the unit circle $|x|=1$ on continua C and our aim is to estimate the distortion of the frontier caused by the mapping. For this purpose, we fix on $|x|=1$ an arc of length α, say $(e^{-i(\alpha/2)}, 1, e^{i(\alpha/2)})$; let C_α be its corresponding image on C. C_α being a continuum, it is permissible to ask about its transfinite diameter δ. The unit circle has the transfinite diameter 1 and since the latter is preserved by mapping functions (30), C has the same. The transfinite diameter being a decreasing functional of the domain, $C_\alpha \subset C$ yields obviously $\delta \leq 1$. It is easily seen that there exist for every α representations bringing δ arbitrarily near to 1. But there arises the question: What is the minimum of δ? The similarity of this problem to the above is clear; in fact our method of solution will be exactly as before.

Let us suppose that δ attains its minimum for a representation (30) which transforms the unit circle into a continuum $\tilde{C}$ and the arc into the subcontinuum $\tilde{C}_\alpha$. We may suppose $\tilde{C}_\alpha$ to be a circle around the origin with radius δ, since an auxiliary representation (30), univalent in the exterior of C_α, may be superposed on the original mapping of $|x|>1$ onto $\tilde{D}$ without changing the normalization (30) or the transfinite diameter δ of $\tilde{C}_\alpha$. If z_0 belongs to $\tilde{C}$, but not to $\tilde{C}_\alpha$, we choose a subcontinuum Γ of $\tilde{C}$ of transfinite diameter ρ which contains z_0, but no point of $\tilde{C}_\alpha$. Again we superpose on the original function, mapping $|x|>1$ on $\tilde{D}$, arbitrary functions (33) which are univalent in the exterior of Γ. We obtain domains D^+ with boundary continuum C^+, containing $C_\alpha{}^+$ as image of the arc. The transfinite diameter δ^+ of $C_\alpha{}^+$ satisfies the inequality $\delta^+ \geq \delta$ in view of the minimum property of δ. Introducing the auxiliary function $\pi(x;y)$, connected with Green's function

of the exterior of $\tilde{C}_a$, we may use (35) for the calculation of $\log \delta^+$. According to our suppposition with resect to $\tilde{C}_a$ we have here too $\pi(x; \infty) = \log x/\delta$. Therefore we find by exactly the same reasoning which led to (37 V) that all points of $\tilde{C}$, not belonging to $\tilde{C}_a$, lie on a radius segment in the exterior of $\tilde{C}_a$. By a rotation we may obtain for $\tilde{C}$ finally the following configuration: It consists of the circle $|x| = \delta$ plus the segment $< -\epsilon, -\delta >$, $(\epsilon > \delta)$. The circle corresponds to the arc $(e^{-i(\alpha/2)}, 1, e^{i(\alpha/2)})$ and the segment to the complementary arc.

Thus, the representation is fixed, except for the still unknown values of δ and ϵ. For their determination we consider the representation

$$\xi = x + \delta^2/x \tag{38}$$

of the exterior of $\tilde{C}$ which has the normalization (30). It transforms $\tilde{C}_a$ into the segment $< -2\delta, 2\delta >$ and the rest of $\tilde{C}$ into the segment $< -\epsilon - \delta^2/\epsilon, -2\delta >$. On the other hand, the function

$$f(x) = x + 1/x + 2\delta - 2 \tag{38'}$$

is univalent for $|x| > 1$ and is of type (30). It transforms the unit circle into a segment such that the arc $(e^{-i(\alpha/2)}, 1, e^{i(\alpha/2)})$ corresponds to the interval $< 2\delta - 2(1 - \cos \alpha/2), 2\delta >$ on the real axis and the complementary arc to the interval $< 2\delta - 4, 2\delta - 2(1 - \cos \alpha/2) >$. In view of the unicity of the mapping function which transforms the said arcs into *different* contiguous segments of the real axis, we get by comparison

$$2\delta - 2(1 - \cos \alpha/2) = -2\delta, \qquad \epsilon + \delta^2/\epsilon = 4 - 2\delta. \tag{38''}$$

In particular, we get from the first equation (38″)

$$\delta = \sin^2 \alpha/4 \tag{39}$$

for the minimum of the transfinite diameter. Thus, we have proved the following distortion theorem:

Every function (30), *univalent in* $|x| > 1$, *maps an arc of the unit circle with aperture* α *on a continuum with transfinite diameter* $d \geq \sin^2 \alpha/4$.

4. If we start with the domain $|x| > r$, every function (30), univalent in this domain, will map an arc with aperture α on a continuum with transfinite diameter $d \geq r \sin^2 \alpha/4$. This leads to an important application: Let C be an arbitrary continuum with transfinite diameter r; then it may be mapped on the circle $|x| = r$ by a function (30), univalent in its exterior.

Divide C into two continua A and B such that they correspond to two complementary arcs on $|x| = r$ with apertures α and $\beta = 2\pi - \alpha$. Then the foregoing theorem yields:

$$d(A) \geq r \sin^2 \alpha/4, \qquad d(B) \geq r \sin^2 \beta/4 = r \cos^2 \alpha/4. \tag{40}$$

Since $r = d(C)$ and $C = A + B$, we get from (40) by addition

$$d(A) + d(B) \geq d(A + B) \tag{41}$$

which establishes in a new way the subadditivity of the transfinite diameter for such a composition (see Schiffer 2).

There are numerous other applications of the theorem of **3** which describes the boundary distortion in case of conformal representation of the unit circle; but we shall not treat them here. We have inserted the proof here only in order to show the use which can be made of (31′).

4. The conformal radii $d_m(y)$.

1. It is obvious that the extremal representation (30′) of D, transforming C_m into a circle of radius δ and all other C_ν into concentric circular arcs, is closely related to the domain function (19), considered in **2**, 2. Suppose, indeed, that $f_m(x; y)$ has in the neighborhood of y the development

$$f_m(x; y) = d_m(y)^{-1}\left[\frac{1}{x - y} + a_0 + a_1(x - y) + \cdots\right], \quad d_m(y) > 0; \tag{42}$$

then $d_m(y) \cdot f_m(x; y)$ is of type (30′) and defines the extremal representation considered. Hence, $d_m(y)$ is the greatest value for the transfinite diameter which C_m can attain by a representation (30′) of D. The area $F(C_m)$, enclosed by the curve C_m may be estimated by means of its transfinite diameter $d(C_m)$ (Pólya 1)

$$F(C_m) \leq \pi d(C_m)^2. \tag{43}$$

Thus, $\pi d_m(y)^2$ is the greatest area, obtainable for C_m by a representation (30′) of D. In virtue of (19) and (42), $d_m(y)$ is connected with $\gamma_m(x; y)$ by

$$\gamma_m(x; y) = \log \frac{1}{|x - y|} + \log \frac{1}{d_m(y)} + O(|x - y|). \tag{44}$$

If $\infty \subset D$, we have

$$\gamma_m(x; \infty) = \log |x| + \log \frac{1}{d_m(\infty)} + O\left(\frac{1}{|x|}\right). \tag{44′}$$

Thus, $d_m(y)$ is related to $\gamma_m(x; y)$ in exactly the same way as is $d(y)$ to

Green's function. In view of the above geometric interpretation, we shall call $d_m(y)$ the conformal radius of C_m with respect to C in y. If we are speaking of $d(C_m; C)$, the conformal radius of C_m with respect to C, we shall refer to $d_m(\infty)$. In order that it be defined, we must have $\infty \subset D$; this will be supposed henceforth in this paragraph.

$d(C_m; C)$ measures the continuum C_m, taking into account all the boundary C. It is invariant with respect to representations (30), univalent in D. It is linearly homogeneous with respect to homotheties, as is easily checked. Introducting (44′) into (27), we get by comparison of coefficients

$$(45) \qquad \delta \log d(C_m; C) = -\frac{1}{2\pi}\int_C \left(\frac{\partial}{\partial n}\gamma_m(z; \infty)\right)^2 \delta n \, ds;$$

introducing (44′) into the variation formula (17), with $\phi(x; y) = \log f_m(x; y)$ in our case, we get

$$(46) \qquad \log d(C^*_m; C^*) = \log d(C_m; C) - R\{e^{2i\phi}\rho^2 f'_m(z_0; \infty)^2 f_m(z_0; \infty)^{-2}\} + o(\rho^2).$$

(45) proves $d(C_m; C)$ to be a decreasing domain function, a fact which follows also easily from its extremal property. Analogously, $d_m(y)$ and even $2\gamma_m(x; y) + \log d_m(x) + \log d_m(y)$ may be shown to be monotonic domain functions; the proof is analogous to that of (32).

2. The following problem requires the application of (46). Divide the boundary C of D into two point sets A and B, each consisting of a finite number of proper continua, and possessing only a finite number of common points. In particular let F be a continuum of A. Consider the conformal radius $d(F; A)$ which is invariant with respect to all representations (30), univalent in the exterior of A. It will change, in general, with respect to mappings (30) of D. We seek the minimum of $d(F; A)$, taking into account all these representations.

In view of the compactness of the family (30) of all functions, univalent in D, there exists at least one function of the family, for which $d(F; A)$ attains its minimum. It maps D, A, F, B on $\tilde{D}$, $\tilde{A}$, $\tilde{F}$, $\tilde{B}$ respectively. It is not uniquely determined since the subsequent superposition of a representation (30), univalent in the exterior of $\tilde{A}$, does not change univalency or normalization of the total mapping function and preserves also $d(\tilde{F}; \tilde{A})$. Thus, it may be supposed that $\tilde{F}$ is a circle around the origin and the other continua of $\tilde{A}$ are concentric circular arcs. In this case, the function $f_F(x; \infty)$ (which corresponds to $f_m(x; \infty)$ in the case $C_m = F$, $C = \tilde{A}$), defining the

canonical representation of $\tilde{A}$ with distinction of $\tilde{F}$, has the form $xd(\tilde{F};\tilde{d})^{-1}$, since $d(\tilde{F};\tilde{A})$ is just the radius of $\tilde{F}$. Now, to find the sought for minimum reduces to the determination of the shape of $\tilde{B}$.

z_0 being a point of $\tilde{B}$ not belonging to $\tilde{A}$, let us consider once more the functions (33), univalent in the exterior of a subcontinuum Γ of $\tilde{B}$, which contains z_0 but no points of $\tilde{A}$. Superposing them on the mapping $D \to \tilde{D}$, we get a normalized univalent representation $D \to D^+$. In view of (46) and the simple particular form of $f_F(x;\infty)$, we have, by an argument similar to that which led to (35′),

$$\log d(F^+;A^+) = \log d(\tilde{F};\tilde{A}) - R\left\{ a\rho^2 \cdot \frac{1}{z_0^2} \right\} + o(\rho^2). \tag{47}$$

Since the minimal property of $d(\tilde{F};\tilde{A})$ involves

$$\log d(F^+;A^+) \geq \log d(\tilde{F};\tilde{A}) \tag{48}$$

for an arbitrary choice of the function (33), the lemma of **3**, 2 yields:

$\tilde{B}$ consists of segments pointing towards the center of $\tilde{F}$.

We have, then, proved the possibility of an interesting type of conformal representation. The boundary C of a domain D may be divided arbitrarily into a finite number of continua, contiguous in a finite number of points only, such that one continuum is mapped on a circle, a given number of continua onto concentric circular arcs and the remainder on radial slits. Every representation of this type solves an extremum problem.

3. The considerations of 2 contain, in particular, the solution of the following problem. Divide the continuum C_1 into the continua A and B which have only two points in common. Let C_A be the aggregate of all points of A and all C_ν $(\nu > 1)$; analogously C_B will consist of B and all C_ν $(\nu > 1)$. Then, we ask for a univalent function (30) in D which imparts to $d(\tilde{A};\tilde{C}_A)$ its minimum, $\tilde{A}$ and $\tilde{C}_A$ corresponding to A and C_A by means of the representation. According to 2 we get the solution:

A conformal representation (30) of D which transforms A into a circle $\tilde{A}$, B into a contiguous radial segment and the remaining C_ν $(\nu > 1)$ into circular arcs, concentric to the circle, yields the minimum value for $d(\tilde{A};\tilde{C}_A)$. Obviously, $d(\tilde{A};\tilde{C}_A)$ is the radius of $\tilde{A}$.

This result becomes interesting when compared with the solution of the following question. Consider all functions (30), univalent in D which transform C_1 into a circle. If r denotes the radius of the circle, α the aperture of

the arc corresponding to A, what is the maximum value of the expression $r \sin^2 \alpha/4$?

Let $\tilde{D}$ be an image of D yielding the maximum; then, $\tilde{C}_1$ is a circle of radius r, and we may suppose that the arc $(e^{-i(\alpha/2)}, 1, e^{i(\alpha/2)})$ corresponds to A. Again we choose a point z_0 on a fixed $\tilde{C}_\nu$ $(\nu > 1)$ and a function (33), univalent in the exterior of a small subcontinuum of $\tilde{C}_\nu$ containing z_0. In this domain, the function

$$(49) \qquad u = \phi(x) - k + \frac{a\rho^2}{z_0} = x + \frac{a\rho^2 x}{(x - z_0)z_0} + o(\rho^2)$$

is also univalent. This representation deforms $|x| = r$, but for ρ sufficiently small the point $r^2/\bar{z}_0$ remains inside the image of this circumference, i. e., lies in the exterior of the image of D, given by (49). Now, we add the further mapping

$$(49') \qquad u^+ = u - \frac{\bar{a}\rho^2 u^2}{(r^2 - \bar{z}_0 u)\bar{z}_0} = x + \frac{a\rho^2 x}{(x - z_0)z_0} - \frac{\bar{a}\rho^2 x^2}{(r^2 - \bar{z}_0 x)\bar{z}_0} + o(\rho^2)$$

which is univalent in $\tilde{D}$ for ρ small enough. Consider the expression

$$(49'') \qquad \psi(x) = x\left[1 + \frac{a\rho^2}{(x - z_0)z_0} - \frac{\bar{a}\rho^2}{(r^2/x - \bar{z}_0)\bar{z}_0}\right];$$

its modulus for $|x| = r$ is $|\psi(x)| = r + o(\rho^2)$. Hence, adding an additional correction term of order $o(\rho^2)$, we infer that $u^+(x)$, so corrected, transforms the circle $|x| = r$ into itself. At the same time, the points $re^{i(\alpha/2)}$ and $re^{-i(\alpha/2)}$ on it are mapped on the points

$$(50) \qquad re^{i(\alpha_{1/2})} = re^{\pm i(\alpha/2)}\left[1 + 2iI\left\{\frac{a\rho^2}{re^{\pm i(\alpha/2)} - z_0)z_0}\right\}\right] + o(\rho^2).$$

Thus, the new arc, corresponding to A, has the aperture

$$(51) \qquad \alpha^+ = \alpha_1 - \alpha_2 = \alpha - 4r \sin \alpha/2 \times R\left\{\frac{a\rho^2}{z_0(r^2 + z_0^2 - 2rz_0 \cos \alpha/2)}\right\} + o(\rho^2).$$

The function (49′), though univalent in $\tilde{D}$, has not yet the normalization (30), but must be divided by $(1 + \frac{\bar{a}\rho^2}{\bar{z}_0^2} + o(\rho^2))$ for this purpose. This operation preserves the aperture α^+, but yields the radius

$$(52) \qquad r^+ = r\left[1 - R\left\{\frac{a\rho^2}{z_0^2}\right\}\right] + o(\rho^2).$$

The expression $r^+ \sin^2 \alpha^+/4$ belongs to a normalized function, univalent in $\tilde{D}$. By (51) and (52) we have

$$(53)\quad r^+ \sin^2 \alpha^+/4 = r \sin^2 \alpha/4 \left[1 - R\left\{ a\rho^2 \frac{(r+z_0)^2}{z_0^2(r^2+z_0^2-2rz_0\cos\alpha/2)} \right\}\right] + o(\rho^2).$$

Since $\tilde{D}$ was supposed to be an extremal domain, we have for every choice of (33)

$$(54)\quad R\left\{ a\rho^2 \frac{(r+z_0)^2}{z_0^2(r^2+z_0^2-2rz_0\cos\alpha/2)} \right\} + o(\rho^2) \geq 0.$$

Applying now the lemma of **3**, 2, we find every $\tilde{C}_\nu$ $(\nu > 1)$ to be an analytic curve $x = x(t)$, satisfying the differential equation

$$(55)\quad \frac{x'(t)^2}{x(t)^2} \frac{[r+x(t)]^2}{[re^{i(\alpha/2)} - x(t)][re^{-i(\alpha/2)} - x(t)]} + 1 = 0.$$

Thus, the extremal domain $\tilde{D}$ is bounded by a circle $\tilde{C}_1$ of radius r and by $n-1$ curves (55). In order to understand better the structure of $\tilde{D}$, consider the function $\zeta(x)$, univalent in the exterior of $\tilde{C}_1$ and of type (30), which maps the arc $\tilde{A}$ of $\tilde{C}_1$, corresponding to A, on a circle around the origin, while its complement $\tilde{B}$ becomes a segment of the negative axis. By these requirements, $\zeta(x)$ is fixed and may be calculated in an elementary way. The radius of the new circle is $r \sin^2 \alpha/4$. The expression $x^2\zeta'(x)^2\zeta(x)^{-2}$ is a regular function for $|x| > r$; it is positive on $\tilde{A}$, negative on $\tilde{B}$, has in $re^{i(\alpha/2)}$ and $re^{-i(\alpha/2)}$ simple poles and in $-r$ a double zero point, as is easily seen from geometric considerations. Taking into account, further, the normalization at infinity, we get by means of Schwarz's principle of reflection

$$(56)\quad \frac{x^2\zeta'(x)^2}{\zeta(x)^2} = \frac{(x+r)^2}{(x-re^{i(\alpha/2)})(x-re^{-i(\alpha/2)})}.$$

With its aid, (55) may be written in simple form, if we put $\zeta[x(t)] = \xi(t)$

$$(55')\quad \frac{\xi'(t)^2}{\xi(t)^2} + 1 = 0;\ \text{i.e.,}\ \xi(t) = \rho_\nu e^{it}, \qquad 0 < \rho_\nu = \text{constant}.$$

The final result may be formulated in the following way:

Among all univalent representations (30) of D which transform A into a circle $\tilde{A}$ and B into a contiguous radial segment, the maximal radius for $\tilde{A}$ is attained, if all other C_ν $(\nu > 1)$ become circular arcs, concentric to $\tilde{A}$.

Thus, the same representation imparts to $d(A; C_A)$ its minimum and to $r \sin^2 \alpha/4$ its maximum which has the value $d(A; C_A)$. Hence, we get the inequality

$$d(A; C_A) \geq r \sin^2 \alpha/4 \tag{57}$$

valid for every representation (30) which transforms C_1 into a circle. By means of such representations, there corresponds to A an arc of aperture α, and to its complement B on C_1 an arc of aperture $\beta = 2\pi - \alpha$. In view of (57) and the analogous formula for B

$$d(B; C_B) \geq r \sin^2 \beta/4 = r \cos^2 \alpha/4, \tag{57'}$$

we get by addition

$$d(A; C_A) + d(B; C_B) \geq r. \tag{58}$$

Since the maximum of r is given by the conformal radius $d(C_1; C)$ (**4**, 1), we get

$$d(A; C_A) + d(B; C_B) \geq d(A + B; C) \tag{58'}$$

in complete analogy to the subadditivity formula (41) for the transfinite diameter.

5. The construction of $\gamma_m(x; y)$ by means of Green's function.

1. The functions $\gamma_m(x; y)$ may be constructed explicitly by means of Green's function $g(x; y)$ and other functions derived from it. In this way, we may derive the variation formula (17) for $\gamma_m(x; y)$ directly from (11). For this construction, we have to study the function $p(x; y)$, introduced in **1**, 2 by (7), more thoroughly. Though its real part $g(x; y)$ is uniform in D, it has with respect to circuits around C_ν the periods

$$2\pi i \omega_\nu(y) = i \int_{C_\nu} \frac{\partial}{\partial n_z} g(y; z)\, ds_z. \tag{59}$$

$\omega_\nu(y)$ is a real-valued harmonic function for $y \subset D$; by virtue of its integral representation it attains on C_ν the value 1, on the remaining C_μ the value 0. $\omega_\nu(y)$ is called the harmonic measure of C_ν in y with respect to D (Nevanlinna 1). Obviously, $\omega_\nu(y)$ is the real part of the analytic function

$$w_\nu(y) = \frac{1}{2\pi} \int_{C_\nu} \frac{\partial}{\partial n_z} p(y; z)\, ds_z. \tag{60}$$

$w_\nu(y)$ is regular in D and possesses periods with respect to circuits around every C_μ; they are

$$2\pi i P_{\mu\nu} = i \int_{C_\mu} \frac{\partial \omega_\nu(y)}{\partial n_y}\, ds_y = \frac{i}{2\pi} \int_{C_\mu} \int_{C_\nu} \frac{\partial^2 g(y; z)}{\partial n_y \partial n_z}\, ds_y ds_z = 2\pi i P_{\nu\mu}. \tag{61}$$

The matrix $(P_{\mu\nu})$ is thus seen to be symmetric.

Consider, for $x \subset D$, the harmonic function

$$v(x) = \sum_{\nu=1}^{n} c_\nu \omega_\nu(x) \tag{62}$$

which has on the curve C_ν the boundary value c_ν. By Green's theorem we have

$$\frac{1}{2\pi}\int_D \int (\operatorname{grad} v)^2 \, d\tau = -\frac{1}{2\pi}\int_C v \frac{\partial v}{\partial n}\, ds = -\sum_{\mu,\nu=1}^{n} P_{\mu\nu} c_\mu c_\nu. \tag{63}$$

This expression is non-negative and vanishes only for grad $v \equiv 0$, i. e., $v(x) = \text{const}$. In this case, all c_ν are necessarily equal, which leads to the theorem:

The quadratic form $\sum_{\mu,\nu=1}^{n} P_{\mu\nu} c_\mu c_\nu$ *is non-positive and vanishes only if all* c_ν *are equal.*

The harmonic function $\sum_{\nu=1}^{n} \omega_\nu(x)$ has everywhere on the boundary of D the value 1 and coincides, therefore, in D with the constant 1. Hence, the analytic function $\sum_{\nu=1}^{n} w_\nu(x)$ is also constant and its period with respect to a circuit around each C_μ vanishes:

$$2\pi i \sum_{\nu=1}^{n} P_{\mu\nu} = 0 \qquad (\mu = 1, \cdots, n). \tag{64}$$

Thus, the sum over every row (or column) in the matrix $(P_{\mu\nu})$ is zero and, consequently, the determinant of this matrix vanishes.

Striking out the n-th row and the n-th column of the matrix $(P_{\mu\nu})$, we obtain a matrix with a corresponding negative-definite form, thus with non-vanishing determinant. Let its inverse matrix be $(p_{\mu\nu}{}^{(n)})$, the upper index indicating that μ and ν do not assume the value n.

2. By means of the previously defined concepts we may easily construct the function $\gamma_n(x; y)$. We consider the analytic function of $x \subset D$, for $y \subset D$ fixed,

$$\phi_n(x; y) = p(x; y) - \sum_{\mu,\nu=1}^{n-1} p_{\mu\nu}{}^{(n)} w_\mu(x) \omega_\nu(y). \tag{65}$$

It has by (59) and (61) the period

$$\pi_\sigma = 2\pi i [\omega_\sigma(y) - \sum_{\mu,\nu=1}^{n-1} p_{\mu\nu}{}^{(n)} P_{\mu\sigma} \omega_\nu(y)] \tag{65'}$$

with respect to a circuit around C_σ. For $\sigma \neq n$, the definition of $(p_{\mu\nu}{}^{(n)})$

insures $\pi_\sigma = 0$. In order to calculate π_n we apply (64) and express $P_{\mu\nu}$ by means of all $P_{\mu\tau}$ $(\tau \neq n)$. It is then easily seen that

$$(65'') \qquad \pi_n = 2\pi i[\omega_n(y) + \sum_{\tau=1}^{n-1} \omega_\tau(y)] = 2\pi i.$$

The function $\phi_n(x; y)$, regular for $x \subset D$ with exception of the logarithmic pole at y, does not change, therefore, for circuits around C_ν $(\nu \neq n)$ and increases by $2\pi i$ for a circuit around C_n. Its real part is constant on every C_ν; in particular, it vanishes on C_n. Hence, $\phi_n(x; y) = \log f_n(x; y)$, $f_n(x; y)$ mapping D onto the exterior of the unit circle, slit along concentric circular arcs. Thus, in view of (19), (59), (60) and (65)

$$(66) \qquad \gamma_n(x; y) = R\{\phi_n(x; y)\} = g(x; y) - \sum_{\mu,\nu=1}^{n-1} p_{\mu\nu}{}^{(n)} \omega_\mu(x) \omega_\nu(y).$$

Analogously, we may construct all the functions $\gamma_m(x; y)$ $(1 \leq m \leq n-1)$, by defining the matrix $(p_{\mu\nu}{}^{(m)})$, inverse to the matrix obtained by striking out in $(P_{\mu\nu})$ the m-th row and the m-th column.

3. If y and z are two fixed points in D, consider the analytic function of x

$$(67) \qquad \phi_n(x; y, z) = \phi_n(x; y) - \phi_n(x; z).$$

It is uniform for every circuit, since the periods for a circuit around C_n cancel each other exactly. It has logarithmic poles in y and z and a constant real part on every boundary continuum C_ν, vanishing in particular on C_n. Thus, it is the logarithm of the function which maps D on a plane, slit along concentric circular arcs, so that y goes to infinity, z to zero and that C_n corresponds to an arc of the unit circle.

This representation is connected with the following extremum problem: Determine a function (30′), univalent in D, which yields the maximum value for $|f'(z)|$, $z \subset D$ (see de Possel, Rengel).

Let $\tilde{D}$ be the domain belonging to the extremal function, $\tilde{C}_\nu$ its boundary curves. Using a point z_0 on $\tilde{C}_\nu$, we consider the functions (33), univalent in the exterior of a subcontinuum Γ of $\tilde{C}_\nu$ with transfinite diameter ρ which contains z_0. Superposing such a function on the extremal representation $D \to \tilde{D}$, gives a resultant conformal representation of type (30′) of D, since

$$(68) \qquad \phi[f(x)] = f^+(x) = f(x) + \frac{a\rho^2}{f(x) - z_0} + o(\rho^2)$$

has at $x = y$ the principal part $\dfrac{1}{x - y}$. Further, we have

$$f^{\nu}(z) = f'(z)\left[1 - \frac{a\rho^2}{(f(z) - z_0)^2} + o(\rho^2)\right]. \tag{69}$$

In view of the maximum property of $|f'(z)|$ there holds for every function (33)

$$R\left\{\frac{a\rho^2}{(f(z) - z_0)^2}\right\} + o(\rho^2) \leq 0. \tag{70}$$

Hence, in virtue of Lemma **3**, 2, every C_ν is an analytic curve $x = x(t)$ with the differential equation

$$x'(t)^2[f(z) - x(t)]^{-2} + 1 = 0. \tag{71}$$

Without restriction of generality we may suppose $f(z) = 0$, since addition of a constant to $f(x)$ changes neither its derivative nor its normalization (30′). Thus, all $\tilde{C}_\nu$ have the form

$$x(t) = c_\nu e^{it}, \qquad 0 < c_\nu = \text{constant}, \tag{70′}$$

i. e., they are all circular arcs around the common center $f(z) = 0$.

Thus, the extremal function $f(x)$, yielding this representation, coincides, by the unicity theorem for this type of conformal representation, with a constant multiple of $\exp\{\phi_n(x; y, z)\}$. Hence,

$$\log|f(x)| = k + g(x; y) - g(x; z) - \sum_{\mu,\nu=1}^{n-1} p_{\mu\nu}^{(n)}\omega_\mu(x)[\omega_\nu(y) - \omega_\nu(z)]. \tag{72}$$

In order to eliminate the constant k, we combine (72) with (29) and (30′); for $x \to y$, we get

$$0 = k + \log\frac{1}{d(y)} - g(y; z) - \sum_{\mu,\nu=1}^{n-1} p_{\mu\nu}^{(n)}\omega_\mu(y)[\omega_\nu(y) - \omega_\nu(z)]. \tag{72′}$$

On the other hand, for $x \to z$, we get in the same way by means of the development $f(x) = f'(z)(x - z) + \cdots$, valid in the neighborhood of the point

$$\log|f'(z)| = k + g(z; y) - \log\frac{1}{d(z)} - \sum_{\mu,\nu=1}^{n-1} p_{\mu\nu}^{(n)}\omega_\mu(z)[\omega_\nu(y) - \omega_\nu(z)]. \tag{72″}$$

By subtracting (72′) from (72″) we get

$$\log|f'(z)| = 2g(z; y) + \log d(z) + \log d(y) + \sum_{\mu,\nu=1}^{n-1} p_{\mu\nu}^{(n)}[\omega_\mu(y) - \omega_\mu(z)][\omega_\nu(y) - \omega_\nu(z)]. \tag{73}$$

The last term is non-positive, since $(p_{\mu\nu}{}^{(n)})$ is the inverse of a matrix with a non-positive quadratic form. The maximum for $|f'(z)|$ is $\geq \frac{1}{|z-y|^2}$, since this value is always attained in the particular case $f(x) = \frac{1}{x-y} - \frac{1}{z-y}$. Thus we have proved for the function (32) the interesting inequality

$$\Gamma(x;y) = 2g(x;y) + \log d(x) + \log d(y) \geq \log \frac{1}{|x-y|^2}. \tag{74}$$

From (66) we obtain, by comparing coefficients

$$\log \frac{1}{d_n(y)} = \log \frac{1}{d(y)} - \sum_{\mu,\nu=1}^{n-1} p_{\mu\nu}{}^{(n)} \omega_\mu(y) \omega_\nu(y). \tag{75}$$

Applying this identity and (66) to (73), we get

$$\log |f'(z)| = 2\gamma_n(z;y) + \log d_n(z) + \log d_n(y). \tag{76}$$

Thus, we have established a simple relation between the domain function $\gamma_n(z;y)$ and the maximal derivative of all functions (30′), univalent in D. It makes obvious the fact, established in **4**, 1, that the right hand side of (76) is a monotone decreasing function of the domain (diminution of the family of competing functions decreases the maximum).

4. After having shown the significance of the expressions $w_\nu(x)$ and $P_{\mu\nu}$, we ask now about their variational formulae. We shall derive them from (11) and get in this way anew the variation formula for $\gamma_n(x;y)$. The variation (2) transforms the domain D with boundary curves C_ν and Green's function $g(x;y)$ into a domain D^* with boundary curves C^*_ν and Green's function $g^*(x;y)$. The connection between Green's functions of both domains is established by (11). If the point x describes a circuit around C_ν, its image point x^* turns around the corresponding C^*_ν, and, by (11) and (59), the period of $p^*(x^*;y^*)$ with respect to this circuit is

$$2\pi i \omega^*_\nu(y^*) = i \int_{C_\nu} \frac{\partial}{\partial n_x} g^*(y^*;x^*)\, ds_x \tag{77}$$

$$= i \int_{C_\nu} \frac{\partial g(y;x)}{\partial n_x}\, ds_x + iR\{e^{2i\phi}\rho^2 p'(z_0;y) \int_{C_\nu} \frac{\partial p'(z_0;x)}{\partial n_x}\, ds_x\} + o(\rho^2)$$

whence, by virtue of (59) and (60),

$$\omega^*_\nu(y^*) = \omega_\nu(y) + R\{e^{2i\phi}\rho^2 p'(z_0;y) w'_\nu(z_0)\} + o(\rho^2). \tag{78}$$

This formula yields the variation of the harmonic measure $\omega_\nu(y)$. Using (61),

we derive herefrom the variation formula for the $P_{\mu\nu}$. In fact, we have from (78):

$$(79)\qquad P^*_{\mu\nu} = \frac{1}{2\pi}\int_{C_\mu} \frac{\partial}{\partial n_y}\,\omega^*_\nu(y^*)\,ds_y$$
$$= P_{\mu\nu} + R\{e^{2i\phi}\rho^2 w'_\nu(z_0)\cdot\frac{1}{2\pi}\int_{C_\mu}\frac{\partial p'(z_0;y)}{\partial n_y}\,ds_y\} + o(\rho^2),$$

whence, in view of (60),

$$(80)\qquad P^*_{\mu\nu} = P_{\mu\nu} + R\{e^{2i\phi}\rho^2 w'_\mu(z_0)w'_\nu(z_0)\} + o(\rho^2).$$

The expressions $p_{\mu\nu}{}^{(n)}$ are defined by the system of equations

$$(81)\qquad \sum_{\lambda=1}^{n-1} p_{\mu\lambda}{}^{(n)}P_{\lambda\sigma} = \delta_{\mu\sigma} = \begin{cases} 1 & \mu = \sigma \\ 0 & \mu \neq \sigma \end{cases} \qquad (\mu,\sigma = 1,\cdots,n-1).$$

We derive from (80) and (81) the following system of equations for $p_{\mu\nu}{}^{(n)*}$:

$$(82)\qquad \delta_{\mu\sigma} = \sum_{\lambda=1}^{n-1} p_{\mu\lambda}{}^{(n)*}P^*_{\lambda\sigma} = \sum_{\lambda=1}^{n-1} p_{\mu\lambda}{}^{(n)*}P_{\lambda\sigma}$$
$$+ R\{e^{2i\phi}\rho^2 w'_\sigma(z_0)\sum_{\lambda=1}^{n-1} p_{\mu\lambda}{}^{(n)}w'_\lambda(z_0)\} + o(\rho^2).$$

Multiplying both sides with $p_{\sigma\nu}{}^{(n)}$ and summing up with respect to σ, we get from (81), in view of the symmetry property $p_{\mu\nu}{}^{(n)} = p_{\nu\mu}{}^{(n)}$

$$(83)\quad p_{\mu\nu}{}^{(n)*} + R\{e^{i2\phi}\rho^2\sum_{\sigma=1}^{n-1} p_{\nu\sigma}{}^{(n)}w'_\sigma(z_0)\cdot\sum_{\lambda=1}^{n-1} p_{\mu\lambda}{}^{(n)}w'_\lambda(z_0)\} + o(\rho^2) = p_{\mu\nu}{}^{(n)},$$

i. e.,

$$(83')\quad p_{\mu\nu}{}^{(n)*} = p_{\nu\mu}{}^{(n)} - R\{e^{2i\phi}\rho^2\sum_{\kappa=1}^{n-1} p_{\mu\kappa}{}^{(n)}w'_\kappa(z_0)\cdot\sum_{\lambda=1}^{n-1} p_{\nu\lambda}{}^{(n)}w'_\lambda(z_0)\} + o(\rho^2).$$

From (66), (11), (78) and (83′) we get finally

$$(84)\quad \gamma^*_n(x^*;y^*) = \gamma_n(x;y)$$
$$+ R\{e^{2i\phi}\rho^2[p'(z_0;x) - \sum_{\mu,\nu=1}^{n-1} p_{\mu\nu}{}^{(n)}w'_\mu(z_0)\omega_\nu(x)][p'(z_0;y)$$
$$- \sum_{\mu,\nu=1}^{n-1} p_{\mu\nu}{}^{(n)}w'_\mu(z_0)\omega_\nu(y)] + o(\rho^2),$$

and using definition (65)

$$(84')\quad \gamma^*_n(x^*;y^*) = \gamma_n(x;y) + R\{e^{2i\phi}\rho^2\phi'_n(z_0;x)\phi'_n(z_0;y)\} + o(\rho^2).$$

This coincides exactly with the variation formula (17) for $\gamma_n(x;y)$.

6. The functionals of a Riemann surface and their variation.

1. The above formal relations between the functionals of a domain D: $g(x;y)$, $\omega_\nu(y)$ and $P_{\mu\nu}$ recall similar connections between the elementary integrals on a Riemann surface and their periods. These analogies are not accidental; they result from the close relations between the theory of conformal representation of multiply connected domains and the theory of Riemann surfaces (see Schottky).

The elementary integrals are functionals of the Riemann surface and we want to study how their variation depends upon that of the surface. To this purpose, we have first to recall certain concepts from the theory of Riemann surfaces.

If the Riemann surface P has the genus $p \geq 1$, there exists on it a canonical system of $2p$ closed rectifiable curves a_j, b_j $(j = 1, \cdots, p)$, such that every pair a_j, b_j has exactly one point of intersection, while no two curves of this system have further common points. Let $v_j(x)$ be analytic on P (i.e., it has a convergent development into power series at every point of P), with the period 1 with respect to a circuit on a_j and the period 0 with respect to circuits on every other $a_\varkappa$. $v_j(x)$ is fixed by this condition, up to an additive constant and is called the j-th elementary integral of first kind on P. The periods of all $v_j(x)$ with respect to circuits on the curves b_k form the matrix (π_{jk}).

Let $t(x;y)$ be an analytic function of x on P, the point y excepted where it has a simple pole with residue 1. Let $t(x;y)$ remain unchanged if x describes any curve a_j. These conditions fix $t(x;y)$ up to an additive constant; $t(x;y)$ is called an elementary integral of the second kind on P.

Let $w(x;y,z)$ be an analytic function of x *on* P, the points y and z excepted where it has logarithmic poles with the principal part $\log \frac{x-z}{x-y}$. Let $w(x;y,z)$ not change if x describes a curve a_j. These conditions fix $w(x;y,z)$ up to an additive constant. $w(x;y,z)$ is called an elementary integral of the third kind on P.

The elementary integrals of the first and second kind are uniform functions of x on the surface $\tilde{\mathrm{P}}$, obtained from P by cutting it along all the curves of the system a_j, b_j. The boundary Σ of P consists of the curves a_j, b_j; but every side of each curve being counted separately, every curve appears twice in Σ. Let $f(x)$ be uniform and meromorphic on $\tilde{\mathrm{P}}$; then

$$S\{f(x)\} = \frac{1}{2\pi i}\int_\Sigma f(x)\,dx \tag{85}$$

represents the sum of all residues of $f(x)$ on $\tilde{\mathrm{P}}$. If, in particular, $f(x)$ is a linear aggregate of a finite number of elementary integrals with coefficients uniform on P, we may, on the other hand, evaluate the expression (85) by using the characteristic periodicity relations of the elementary integrals, mentioned above. Riemann's method of boundary integration compares both results and obtains in this way relations between elementary integrals and their periods.

Since we shall use the result later on, we apply Riemann's method to

$$(86) \qquad S\{t(x;u)w'(x;y,z)\} = \frac{1}{2\pi i}\int_{\Sigma} t(x;u)w'(x;y,z)\,dx, \qquad u \subset \tilde{\mathrm{P}}$$

(the dash denotes as usual the derivative with respect to the first variable). Integrating along Σ, we have to pass every curve a_j and b_j twice, but in opposite directions. We get from one bank of the curve a_j to the other by following the curve b_j, and from one bank of b_j to the other by describing a_j (starting always in their common point). $t(x;u)$ having the period zero with respect to every circuit a_j, it has the same value on both banks of every b_j; hence, the integrations along the two banks of b_j in (86) cancel each other. The values of $t(x;u)$ on both banks of a_j differ by a constant period, depending, however, upon u, say $p_j(u)$. Thus

$$(86') \qquad S\{t(x;u)w'(x;y,z)\} = -\sum_{k=1}^{p}\frac{1}{2\pi i}\int_{a_k} w'(x;y,z)\,dx \cdot p_k(u).$$

Now, $w(x;y,z)$ does not change after describing a full circuit a_k; hence, each integral in (86') vanishes, and their sum also.

On the other hand, we have, by virtue of the residue theorem,

$$(87) \qquad S\{t(x;u)w'(x;y,z)\} = w'(u;y,z) - [t(y;u) - t(z;u)].$$

This being zero by the foregoing argument, we have established the following relation between elementary integrals of the second and the third kind:

$$(88) \qquad w'(x;y,z) = t(y;x) - t(z;x).$$

In the same way the following relations are proved: The equalities

$$(89) \qquad \pi_{jk} = \pi_{kj};$$

the periodicity relations

$$(90) \qquad \frac{1}{2\pi i}\int_{b_j} t'(x;y)\,dx = -v'_j(y),$$

$$(91) \qquad \frac{1}{2\pi i}\int_{b_j} w'(x;y,z)\,dx = -[v_j(y) - v_j(z)],$$

and the symmetry relations

$$t'(x;y) = t'(y;x) \tag{92}$$

$$w(x;y,z) - w(u;y,z) = w(y;x,u) - w(z;x,u). \tag{93}$$

2. After these preparations let us examine the change of the elementary integrals due to a variation of P. It is known that they depend continuously upon P (see Ritter, Koebe 2). But no explicit formula, describing the dependency, has been given as yet. We consider the following particular variations of P.

Let $f(x)$ be uniform and meromorphic on P with simple poles z_ν $(\nu = 1, \cdots, N)$ and residues r_ν at z_ν. For sake of simplicity we suppose that no z_ν coincides with a branch point of P or lies on a curve of our canonical system. Consider the function

$$x^* = x + \rho^2 f(x). \tag{94}$$

Fix around each z_ν a small circumference k_ν, not containing any branch point of P or a point of the canonical curve system, such that no two circles overlap. Discarding the interiors of k_ν from P, we obtain a surface P_0 with N holes. If ρ is sufficiently small, P_0 is mapped by (94) in a one-to-one manner on a surface P^*_0 with N holes, which can be completed to a closed Riemann surface P* (see Schiffer 3).

Our purpose is to calculate for this variation $P \to P^*$ the corresponding variation formulae for the elementary integrals and their periods. We remark that P and P* are of the same genus and that the variation (94) transforms the system a_j, b_j again into a canonical system a^*_j, b^*_j on P*. The functions $v^*_j(x)$, $t^*(x;y)$, $w^*(x;y,z)$ being the elementary integrals on P*, the functions $v^*_j(x^*(x))$, $t^*(x^*(x);y^*(y))$ and $w^*(x^*(x);y^*(y),z^*(z))$ are given on P_0 by means of (94). If x describes the system a_j, b_j, its image x^* does the same with respect to a^*_j, b^*_j. We establish now the equality

$$\frac{1}{2\pi i}\int_\Sigma t(x;u)\frac{d}{dx}w^*(x^*;y^*,z^*)\,dx = \sum_{j=1}^{p} v'_j(u)\int_{a_j}\frac{d}{dx}w^*(x^*;y^*,z^*)dx \tag{95}$$

by means of the periodicity properties of $t(x;u)$. In fact, $t(x;u)$ has the periods zero with respect to the circuits a_j and, by (9ι , the periods $-2\pi i\, v'_j(u)$ regarding the b_j. Further, $w^*(x^*;j^*,z^*)$ does not change, if x^* makes a full circuit around a^*_j, i. e., if x describes a_j. Thus,

$$\frac{1}{2\pi i}\int_\Sigma t(x;u)\frac{d}{dx}w^*(x^*;y^*,z^*)\,dx = 0. \tag{96}$$

If, on the other hand, u, y, z lie in P_0, we may evaluate the last integral by Cauchy's theorem and so derive, from (96), the equality

$$(97)\qquad 0 = \frac{d}{du}\, w^*(u^*; y^*, z^*) - [t(y; u) - t(z; u)] + \sum_{\nu=1}^{N} \frac{1}{2\pi i} \int_{k_\nu} t(x; u)\, \frac{d}{dx}\, w^*(x^*; y^*, z^*)\, dx.$$

By elementary development into series, using (94) and the residue theorem, we find

$$(98)\qquad \frac{1}{2\pi i} \int_{k_\nu} t(x; u)\, \frac{d}{dx}\, w^*(x^*; y^*, z^*)\, dx = -\, r_\nu \rho^2 t'(z_\nu; u)\, w'(z_\nu; y, z) + o(\rho^2).$$

In view of (88) and (98) we may write, instead of (97),

$$(99)\qquad \frac{d}{du}\, w^*(u^*; y^*, z^*) = \frac{d}{du}\, w(u; y, z) + \sum_{\nu=1}^{N} r_\nu \rho^2 t'(z_\nu; u)\, w'(z_\nu; y, z) + o(\rho^2).$$

By virtue of (92), (99) may be integrated with respect to u from x_0 to x. Applying once more (88), we find

$$(100)\qquad w^*(x^*; y^*, z^*) - w^*(x_0^*; y^*, z^*) = w(x; y, z) - w(x_0; y, z) + \sum_{\nu=1}^{N} r_\nu \rho^2 w'(z_\nu; x, x_0)\, w'(z_\nu; y, z) + o(\rho^2).$$

Now, let x describe the curve b_j; then x^* will describe b^*_j and, in view of (91), (88) and (90), we get by comparison of the periods on both sides of (100):

$$(101)\qquad v^*_j(y^*) - v^*_j(z^*) = v_j(y) - v_j(z) + \sum_{\nu=1}^{N} r_\nu \rho^2 v'_j(z_\nu)\, w'(z_\nu; y, z) + o(\rho^2).$$

Next, let y describe the curve b_k and compare once more the increments on both sides:

$$(102)\qquad \pi^*_{jk} = \pi_{jk} - \sum_{\nu=1}^{N} r_\nu \rho^2 v'_j(z_\nu)\, v'_k(z_\nu) + o(\rho^2).$$

The great similarity between (100), (101) and (102), on the one hand, and (11), (72) and (80), on the other, is obvious. The above formulas determine explicitly the variation of the elementary integrals and their periods on Riemann surfaces. To their applications on extremum problems concerning Riemann surfaces we hope to return elsewhere.

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EULER-KNOPP SUMMABILITY OF CLASSES OF CONVERGENT SERIES.*

By M. S. MACPHAIL.

In connection with the study of Euler-Knopp methods of summability recently made by R. P. Agnew [1], the theorem of the present note may be of interest. A given series $u_0 + u_1 + \cdots$ with partial sums $s_n = u_0 + u_1 + \cdots + u_n$ is summable $E(r)$ to σ if $\sigma_n \to \sigma$ as $n \to \infty$, where

$$E(r): \qquad \sigma_n = \sigma_n(r) = \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} s_k,$$

r being real or complex. In case Σu_n converges, we denote its sum by s. It is well known that $\sigma_n \to s$ for every convergent series, if and only if r lies in the interval $0 < r < 1$; the theorem of this note shows that $\sigma_n \to s$ for specified classes of convergent series, if and only if r lies in correspondingly larger regions of the complex plane.

THEOREM.[1] *If $R > 1$, a necessary and sufficient condition for $E(r)$ to have the property that Σu_n is summable $E(r)$ to s whenever $\Sigma u_n z^n$ has its radius of convergence greater than or equal to R, is that*

$$|r/R| + |1-r| < 1. \tag{1}$$

Proof of sufficiency. Assume that (1) holds, and let $\Sigma u_n z^n$ have radius of convergence $R > 1$, so that Σu_n converges and s is defined. Choose R_1 such that $1 < R_1 < R$, and

$$|r/R_1| + |1-r| < 1. \tag{2}$$

Then

$$|s - s_n| = \Big|\sum_{k=n+1}^{\infty} u_k\Big| = \Big|\sum_{k=n+1}^{\infty} (u_k R_1^k) R_1^{-k}\Big| \leqq M_1 \sum_{k=n+1}^{\infty} R_1^{-k} = M_2 R_1^{-n},$$

* Received September 17, 1945; Revised November 26, 1945.

[1] The author wishes to thank Professor R. P. Agnew for the very simple proof here given.

where $M_1 = \max_{k \geqq 0} |u_k R_1^k|$ and $M_2 = M_1/(R_1 - 1)$. Hence

$$\begin{aligned} |s - \sigma_n| &= \left|\sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k}(s - s_k)\right| \\ &\leqq \sum_{k=0}^{n} \binom{n}{k} |r|^k |1-r|^{n-k} M_2 R_1^{-k} \\ &= M_2(|r/R_1| + |1-r|)^n, \end{aligned}$$

and therefore (2) implies that $\sigma_n \to s$.

Proof of necessity. Take $u_n = (\theta R)^{-n}$, where $R > 1$ and θ is a complex number, for which $|\theta| = 1$, to be determined later. Then $\Sigma u_n z^n$ has radius of convergence R, and $s = \Sigma u_n = \theta R/(\theta R - 1)$. Also,

$$\begin{aligned} \sigma_n &= \frac{1}{\theta R - 1} \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k}[\theta R - (\theta R)^{-k}] \\ &= \frac{\theta R}{\theta R - 1} - \frac{[(r/\theta R) + (1-r)]^n}{\theta R - 1}. \end{aligned}$$

If now we assume that Σu_n is summable $E(r)$ to s, i. e., that $\sigma_n \to s$, it follows that

$$|(r/\theta R) + (1-r)| < 1,$$

from which (1) is obtained by choosing θ so that $|\theta| = 1$ and $\arg(r/\theta R) = \arg(1-r)$.

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AN ABELIAN LEMMA CONCERNING ASYMPTOTIC EQUILIBRIA.*

By AUREL WINTNER.

The original purpose of the following considerations has been a *purely Abelian* approach to the principal limit theorem of [1]. Such an approach requires the elimination of the explicit machinery of the successive approximations, which underly the proof given in [1].

It turns out that such a simplification of the method not only is possible but, corresponding to its more primitive nature, is such as to lead to generalizations which are not within the scope of successive approximations. In fact, the only restriction to be imposed on the function $f = f(t;x)$ defining the (vectorial) differential equation $x' = f(t;x)$ will be a restriction *in the large,* whereas the convergence of the process of successive approximations can be assured only by *local* assumptions (as exemplified by Lipschitz's sufficient condition). In [1], asumptions of *both* kinds were needed. In the sequel, even the *single* assumption to be made, the assumption in the large, is more general than the corresponding assumption of [1].

Another consequence of the omission of any restriction of a local nature will be the inclusion of differential equations in which the solutions are not uniquely determined by the initial conditions. In fact, the question of uniqueness is just as much a purely local affair as the (local) convergence of the process of successive aproximaptions.

Let $f = (f_1, \cdots, f_n)$ and $x = (x_1, \cdots, x_n)$ be vectors with real components f_i, x_i, and let $|u|$ denote the Euclidean length of the vector $u = (u_1, \cdots, u_n)$. Then the generalized Abelian lemma to be proved can be formulated as follows:

Let f be a continuous function of the position $(t;x)$ on the product space of the half-line $0 \leqq t < \infty$ and of the whole Euclidean x-space. Suppose that there exists a pair of functions λ, ϕ satisfying

$$|f(t;x)| < \lambda(t)\phi(|x|), \tag{1}$$

where $\lambda(r)$ and $\phi(r)$ are positive when $0 \leqq r < \infty$, continuous when $0 < r < \infty$, and are subject to

* Received March 22, 1946.

$$\int^{\infty} \lambda(t)\,dt < \infty \tag{2}$$

and

$$\int^{\infty} (dr)/\phi(r) = \infty. \tag{3}$$

Then, if x_0 is any point of the x-space, and if

$$x = x(t), \qquad 0 \leqq t < \epsilon = \epsilon(x_0) \tag{4}$$

is any solution vector of

$$x' = f(t;x), \qquad x(0) = x_0, \qquad (x' = dx/dt), \tag{5}$$

it is possible to extend (4) *to a solution of* (5) *over the whole half-line $0 \leqq t < \infty$, and every solution vector*

$$x = x(t), \qquad 0 \leqq t < \infty \tag{6}$$

is such as to tend to a (finite) limit vector, $x(\infty)$, as $t \to \infty$.

This is the more interesting as (for reasons pointed out above) a solution (6) of $x' = f(t;x)$ need not be uniquely determined by the initial condition $x(0) = x_0$. As a matter of fact, the integration constant $x(\infty)$ (which is claimed to exist for every $x(0)$ and *for every choice of* (6) *when $x(0)$ is fixed*) need not be a single-valued function of $x(0)$.

Since (1) remains fulfilled if λ, ϕ are replaced by functions λ^*, ϕ^* satisfying $\lambda \leqq \lambda^*$, $\phi \leqq \phi^*$, and since (2), (3) are restrictions on λ, ϕ only when t, r are large, it is clear that, without violating (2) and (3), it can be assumed that

$$\limsup_{t \to +0} \lambda(t) < \infty \tag{7}$$

and

$$\liminf_{r \to +0} \phi(r) \neq 0, \qquad (\text{and } \phi(0) \neq 0). \tag{8}$$

The positive function $\lambda(t)$ is supposed to be continuous when $0 < t < \infty$ (actually, (2) alone would suffice). In view of (7), this implies that, if $0 < T < \infty$, there exists a constant $\lambda = \lambda_T$ satisfying $\lambda(t) < \lambda_T$ when $0 \leqq t \leqq T$. It follows therefore from (1) and (3) that $\phi_T(r) = \lambda_T \phi(r)$ is a continuous function having the following properties:

$$|f(t;x)| < \phi_T(|x|) \text{ if } 0 \leqq t \leqq T$$

and

$$\int^{\infty} (dr)/\phi_T(r) = \infty.$$

But these properties imply that every solution vector (4) of $x' = f(t; x)$ can be continued over the whole interval $0 \leqq t \leqq T$ (cf. [2], Appendix). Since T can be chosen arbitrarily large, it follows that every solution (4) can be extended into a solution (6). Hence, only the assertion following (6), that is, the existence of $x(\infty)$, remains to be proved.

To this end, let (6) be any solution of $x' = f(t; x)$. Then, from (1),

$$\int_u^v |\, dx(t)\,|/\phi(|\, x(t)\,|) < \int_u^v \lambda(t)\, dt, \tag{9}$$

if $0 < u < v < \infty$. On the other hand, since $|\, x(t)\,|$ and $\phi(r)$ are real-valued and continuous, and since a real-valued, continuous function cannot leave out a value, it is clear that

$$\int_\alpha^\beta (dr)/\phi(r) \leqq \int_u^v |\, dx(t)\,|/\phi(|\, x(t)\,|), \tag{10}$$

where

$$\alpha = \alpha(u, v) = \min_{u \leqq t \leqq v} |\, x(t)\,|, \qquad \beta = \beta(u, v) = \max_{u \leqq t \leqq v} |\, x(t)\,|. \tag{11}$$

Finally, it is seen from (9) and (1) that there exists a t_0 having the property that

$$\int_u^v |\, dx(t)\,|/\phi(|\, x(t)\,|) < 1 \text{ if } v > u > t_0. \tag{12}$$

According to (10) and (12), the functions (11) satisfy the inequality

$$\int_\alpha^\beta (dr)/\phi(r) < 1 \text{ if } v > u > t_0.$$

In view of (3), this inequality implies that both functions (11) remain bounded as $u \to \infty$ (hence $v \to \infty$). In order to see this, it is sufficient to let v tend to ∞ while u $(> t_1)$ is fixed, and to observe that both functions (11) are monotone in u and in v.

Accordingly, the second of the functions (11) is bounded on the range

$0 < u < v < \infty$. This means that there exists a constant, say M, satisfying $|x(t)| < M$, where $0 \leqq t < \infty$. On the other hand, (8) and the continuity of the positive function $\phi(r)$ on the half-line $0 < r < \infty$ imply the existence of a positive constant, say m, satisfying $\phi(r) > m$, where $0 \leqq r < M$. Hence it is seen from (12) that

$$\int_u^v |dx(t)| < \text{const. if } v > u > t_0.$$

Accordingly, *the solution vector* (6) *is of bounded variation,*

$$\int^{\infty} |dx(t)| < \infty.$$

The statement made after (6), namely, the existence of a limiting position for the vector $x(t)$ as $t \to \infty$, is just a corollary of this fact.

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COMPLEX SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS.*

By HERMAN CHERNOFF.

1. Introduction. In order to study solutions of partial differential equations

$$(1.1) \qquad L(u) \equiv \Delta u + A(x,y)u_x + B(x,y)u_y + C(x,y)u = 0$$

Bergman considered "classes" of complex solutions of (1.1). Some of these "classes" have useful properties; their introduction frequently helps in the investigation of real solutions in a manner which is analogous to the way analytic functions of a complex variable help in the study of real solutions of the Laplace equation $\Delta u = 0$ (see Bergman, [1], [2], [3]).[1]

Let $E(z, \bar{z}, t)$ [2] denote a complex function of the three real variables x, y and t which is defined in a domain of the (x, y)-plane containing the origin and for $-1 \leq t \leq 1$.

The totality of functions

$$(1.2) \qquad u(z,\bar{z}) = P[f] \equiv \int_{-1}^{1} E(z,\bar{z},t) f[(z/2)(1-t^2)] \frac{dt}{(1-t^2)^{1/2}}$$

which we obtain when $f(\zeta)$ ranges over the totality of analytic functions of a complex variable which are regular at the origin is denoted as the "class" $\mathfrak{B}(E)$; E is called the generating function of the class and f is the associate of $u(z, \bar{z})$ with respect to the operator P.

If $E(z, \bar{z}, t)$ is a particular solution of a certain differential equation associated with (1.1), then every function $u(z, \bar{z})$ in $\mathfrak{B}(E)$ satisfies (1.1) and thus $\mathfrak{B}(E)$ is a "class" of complex solutions of (1.1) (See [1], [2], [3]). Bergman has shown that to every equation (1.1), where A, B, and C are entire functions of x and y, there exists at least one $E(z, \bar{z}, t)$ such that the class $\mathfrak{B}(E)$ has the following property: If $U(x, y)$ is a real solution of (1.1) at the origin then there exists a function $u(z, \bar{z})$ in $\mathfrak{B}(E)$ such that

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[1] Numbers in brackets refer to bibliography.

[2] The notation $g(z, \bar{z})$ will denote $g_1(x, y) + ig_2(x, y)$ where $z = x + iy$, $\bar{z} = x - iy$ and g_1, g_2, x and y are all real.

$$U(x, y) = Re[u(z, \bar{z})].$$

This is the basic relation between the solutions of the Laplace equation $\Delta u = 0$ and the class $\mathcal{C}(1)$ which is the class of analytic functions regular at the origin.

Various questions arise in connection with the introduction of classes of functions. One of them is the problem of the distribution of the b-points of a function $u(z, \bar{z})$ of the class $\mathcal{C}(E)$. Just as in the classical theory of analytic functions of a complex variable this problem consists mainly of relations between the behavior of

$$m[r, u(z, \bar{z})] = (1/2\pi) \int_0^{2\pi} \log^+ | u(re^{i\phi}, re^{-i\phi}) | \, d\phi \,^3$$

and

$$n[r, (u(z, \bar{z}) - b)^{-1}] = (1/2\pi i) \int_{|z|=r} d \log [u(z, \bar{z}) - b]$$

for $r \to \infty$. In addition the relations between these quantities and the coefficients of the series expansion of $u(z, \bar{z})$ at the origin is of importance.

If $u(z, \bar{z}) \neq b$ on $| z | = r$, $n[r, (u(z, \bar{z}) - b)^{-1}]$ represents the "number" of b-points of the function $u(z, \bar{z})$ in $| z | \leq r$. (A more detailed explanation of what is to be understood by "number" will be given in the next section.)

We note further that Bergman showed (See [3, § 8]) that when the E function is of a certain form, it is possible to obtain upper bounds for the growth of $n[r, (u(z, \bar{z}) - b)^{-1}]$ in terms of the coefficients $\{A_{mo}\}$ of the expansion

$$u(z, \bar{z}) = \sum_{m,n=0}^{\infty} A_{mn} z^m \bar{z}^n.$$

It is the object of this paper to derive an upper bound for $m[r, u(z, \bar{z})]$ in terms of $\sum_{\nu=1}^{q} n[r, (u - a_\nu)^{-1}]$, and other quantities. This inequality is analogous to the inequality of the Second Fundamental Theorem of Nevanlinna for meromorphic functions.

These considerations suggest the extension of the Picard Theorem [4] to complex solutions of elliptic equations. It will be seen that under certain conditions an analogue can be established.

In Section **6**, certain results concerning coefficients of the expansion of

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[3] $\log^+ | a | = \log | a |$ if $| a | \geq 1$
$\log^+ | a | = 0$ if $| a | \leq 1$

[4] In one form the Picard Theorem states that an entire function assumes all finite values with only one possible exception.

"pseudo-simple" functions are derived. These results represent a generalization of classical theorems concerning the coefficients of the expansion of simple (univalent) functions.

2. Generating Functions of a Special Form. In this section we shall describe the general idea of the treatment of certain classes of "entire functions" developed by Bergman in [3] and we shall formulate some previous results which will be used in what follows. Furthermore we shall discuss in detail the properties of certain notions which are of importance in connection with the study of the distribution of b-points.

In Bergman's approach the class of functions

$$u(z,\bar{z}) = \int_{-1}^{1} E(z,\bar{z},t) f[(z/2)(1-t^2)] \frac{dt}{(1-t^2)^{1/2}} \tag{2.1}$$

is of special interest when there exists a function $E^{(1)}(z,s,t)$ which is an entire function of the two complex variables z and s, with the property that if $|z| = s$, then

$$E(z,\bar{z},t) = E^{(1)}(z,s,t). \tag{2.2}$$

If, in addition, we impose the condition that $f(z)$ be entire. we have a class of entire solutions of our elliptic equation such that for $|z| = s$

$$u(z,\bar{z}) = v_s(z) \equiv \int_{-1}^{1} E^{(1)}(z,s,t) f[(z/2(1-t^2)] \frac{dt}{(1-t^2)^{1/2}} \tag{2.3}$$

and $v_s(z)$ is an entire function of z for each value of s. This means that on the circle $|z| = s$, $u(z,\bar{z})$ takes on the values of an entire function of z. This property will be the one through which we shall be able to apply classical theorems of analytic functions of a complex variable to theorems concerning these classes of complex solutions of elliptic equations.

An example of such a class is the class of solutions of $\Delta u + u = 0$ where $E(z,\bar{z},t) = e^{it(z\bar{z})^{\frac{1}{2}}}$ and hence $E^{(1)}(z,s,t) = e^{ist}$. Thus for $|z| = s$, and $f(z) = \sum_0^\infty a_n z^n$

$$u(z,\bar{z}) = v_s(z) = \int_{-1}^{1} e^{ist} \sum_0^\infty a_n [(z/2)(1-t^2)]^n \frac{dt}{(1-t^2)^{1/2}} = \sum_0^\infty a_n c_n \frac{J_n(s)}{s^n} z^n$$

where $J_n(s)$ is the Bessel Function of order n and c_n are positive constants. We shall return to this example later.

In general if $v_s(z) = \Sigma c_{mn} z^m s^n$, $u(z,\bar{z}) = \Sigma A_{mn} z^m \bar{z}^n$, $(m > n)$, $f(z) = \Sigma a_n z^n$ and $E^{(1)}(z,s,t) = \Sigma E_\mu^{(m)}(t) s^m z^\mu$, then the following relations have been established by Bergman (See [3, § 7])

$$(2.4) \qquad c_{mn} = \sum_{\nu=0}^{n} \frac{A_{\nu 0}\Gamma(\nu+1)}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})} \int_{-1}^{1} E_{n-\nu}{}^{(m)}(t)[1-t^2]^{\nu-\frac{1}{2}}dt$$

$$(2.5) \qquad A_{n0} = \frac{a_n \pi^{\frac{1}{2}}\Gamma(n+\frac{1}{2})}{2^n\Gamma(n+1)}.$$

In order to study the distribution of b-points of a function $u(z, \bar{z})$ the following analogue is considered. For an analytic function of a complex variable, $f(z)$, the number of b-points in a region bounded by a set of Jordan curves C on which there are no b-points is

$$d = (1/2\pi i) \int_C d \log [f(z) - b].$$

This expression also accounts for the multiplicity of the b-points. Analogously, we may consider

$$(2.6) \qquad d = (1/2\pi i) \int_C d \log [u(z, \bar{z}) - b]$$

to characterize the region with respect to the value b of $u(z, \bar{z})$ if $u(z, \bar{z})$ is continuously differentiable along the boundary and continuous in the region (See [5]).

The following properties can be proved, if $u(z, \bar{z})$ is continuously differentiable in x and y.

(i) d is always an integer, either negative, positive or zero. For functions analytic in the domain, however, d is always greater than or equal to zero.

(ii) If d_1 characterizes the region R_1 with respect to the b values of $u(z, \bar{z})$, d_2 characterizes the region R_2 with respect to the b values of $u(z, \bar{z})$, and R_1 and R_2 have no inner points in common, then $d_1 + d_2$ characterizes the region $R_3 = R_1 \cup R_2$ with respect to the b values of $u(z, \bar{z})$.

(iii) If d is not zero then $u(z, \bar{z})$ assumes the value b at least once in the region.

(iv) d is a characteristic of the b-points in the region and does not depend upon the region itself. That is to say, if we deform the region so that no b-points are omitted or added, d remains fixed. Thus every *set of b-points* which may be isolated from other b-points by a region, whose boundary consists of a set of Jordan curves, has an *index*. In particular, an isolated b-point has an index and for an analytic function of a complex variable, the index of a b-point is the multiplicity of the value b at the point.

We may now define

$$(2.6) \qquad n[s,(u(z, \bar{z}) - b)^{-1}] = (1/2\pi i) \int_{|z|=s} d \log [u(z, \bar{z}) - b].$$

In the classical theory of functions of a complex variable, if $f(z)$ is analytic in $|z| \leq s$, $n[s,(f(z)-b)^{-1}]$ is defined as the number (counting multiplicity) of b-points of $f(z)$ in $|z| \leq s$ if $f(z) \not\equiv b$. Our definition coincides with this if $f(z)$ has no b-points on $|z| = s$. If $f(z)$ has b-points on $|z| = s$ our definition does not apply.[5] In the following we shall write $\tilde{n}[s,(f(z)-b)^{-1}]$ to stand for the number of b-points (counting multiplicity) of the analytic function $f(z)$ in $|z| \leq s$.

For the class of functions in which we are interested $u(z,\bar{z}) = v_s(z)$ for $|z| = s$. Thus, if $n[s,(u(z,\bar{z})-b)^{-1}]$ is defined,

$$(2.7) \qquad n[s,(u(z,\bar{z})-b)^{-1}] = n[s,(v_s(z)-b)^{-1}] = \tilde{n}[s,(v_s(z)-b)^{-1}].$$

Making use of the fact that $\tilde{n} \geq 0$, Bergman proved (See [3, § 8]) that, if it exists,

$$(2.8) \qquad n[s,(u(z,\bar{z})-b)^{-1}] \geq 0.$$

Referring to the example $\Delta u + u = 0$, we had, if $f(z) = z^m$,

$$v_s(z) = c_m \frac{J_m(s)}{s^m} z^m.$$

If $b = 0$, it is evident that if $J_m(s) \neq 0$, $n[s,(u(z,\bar{z})-b)^{-1}]$ is defined and thus

$$n[s,u(z,\bar{z})^{-1}] = \tilde{n}[s,(c_m \frac{J_m(s)}{s^m} z^m)^{-1}] = m.$$

This means that the index of the zero-point at the origin is m. Now, $u(z,\bar{z})$ is also zero on the set of circles of radius s_ν where $J_m(s_\nu) = 0$. The index of each of these circles is zero for, if we consider for the region isolating the circle $|z| = s_\nu$ from all other zero points the ring bounded by $|z| = s_\nu - \epsilon$ and $|z| = s_\nu + \epsilon$, ϵ small, then the index of the circle is $n[s+\epsilon, u(z,\bar{z})^{-1}] - n[s-\epsilon, u(z,\bar{z})^{-1}] = 0$.[6] If $b \neq 0$, we have

$$n[s,(u(z,\bar{z})-b)^{-1}] = m \qquad \text{if } |b| < |c_m J_m(s)|$$
$$n[s,(u(z,\bar{z})-b)^{-1}] = 0 \qquad \text{if } |b| > |c_m J_m(s)|.$$

On each circle for which $|b| = |c_m J_m(s)|$, there are m b-points and it can be shown without trouble that the index of each b-point is $+1$ or -1. Also, on the first such circle the indices are each $+1$, on the second -1, on the

[5] It is noteworthy that if $f(z)$ is analytic and has b-points on $|z| = s$, then although our definition does not apply, the integral of our definition will give the number of b-points in $|z| < s$ plus half the number on $|z| = s$ (counting multiplicity).

[6] The convention of integrating in the direction such that the region is to the left must be observed and gives the negative sign to $n[s-\epsilon, u(z,z)^{-1}]$.

third $+1$, alternating in this manner until $J_m(s)$ becomes so small that there are no more of these circles.

Geometrically speaking, if an isolated b-point has an index of -1, it means that as a small curve is drawn counter clockwise about the b-point in the z plane, the corresponding $u(z, \bar{z})$ points describe a curve drawn clockwise about b in the u plane. An index of -2 would mean that the $u(z, \bar{z})$ points describe a path which circles $u = b$ twice in a clockwise direction.

In addition, a concept used in the theory of meromorphic functions is very easily generalized. If $f(z)$ is a meromorphic function then there is defined

$$m[s, f(z)] = (1/2\pi) \int_0^{2\pi} \log^+ | f(se^{i\theta}) | \, d\theta.$$

Analogously, given a function $g(z, \bar{z})$, one defines (See Bergman [3, § 8])

$$(2.9) \qquad m[s, g(z, \bar{z})] = (1/2\pi) \int_0^{2\pi} \log^+ | g(se^{i\theta}, se^{-i\theta}) | \, d\theta$$

if the integral exists. If $g(z, \bar{z}) = f(z)$ on $| z | = s$ where $f(z)$ is meromorphic, then $m[s, g(z, \bar{z})]$ exists. Thus $m[s, u(z, \bar{z})]$ exists and if $u(z, \bar{z}) \not\equiv b$ on $| z | = s$, then $m[s, (u(z, \bar{z}) - b)^{-1}]$ exists too. In addition

$$(2.10) \quad m[s, u(z, \bar{z})] = m[s, v_s(z)]; \quad m[s, (u(z, \bar{z}) - b)^{-1}] = m[s, (v_s(z) - b)$$

We may consider $m[s, (u(z, z) - b)^{-1}]$ to be a measure of the intensity of the values b near the circle $| z | = s$. Furthermore, since $u(z, \bar{z})$ is always finite, $m[s, u(z, \bar{z})]$ may be considered as a measure of the growth of the function $u(z, \bar{z})$.

Using the first Theorem of Nevanlinna, Bergman derived the following bounds (see [3, § 8]):
If $u(z, \bar{z}) = \Sigma A_{mn} z^m \bar{z}^n$ and

$$\rho = \lim_{m+n \to \infty} \sup \frac{(m+n) \log (m+n)}{\log \left| 1 / \sum_{\nu=0}^{n} \frac{A_{\nu 0} \Gamma(\nu+1)}{\pi^{1/2} \Gamma(\nu+\frac{1}{2})} \int_{-1}^{1} E_{n-\nu}^{(m)}(t) (1-t^2)^{\nu-\frac{1}{2}} dt \right|} < \infty$$

then for arbitrary $\epsilon > 0$

(a) $\max u(z, \bar{z}) < c_1 e^{s^{\rho+\epsilon}}$, $\therefore m[s, u(z, \bar{z})] \leq s^{\rho+\epsilon}$ for s large enough

(b) $n[s, (u(z, \bar{z}) - b)^{-1}] < c_2 s^{\rho+\epsilon}$ if $| v_s(0) - b | \geq c > 0$ where c is an arbitrary fixed constant.

(c) $m[s, (u(z, \bar{z}) - b)^{-1}] < c_3 s^{\rho+\epsilon}$ if $| v_s(0) - b | \geq c$.

3. An Upper Bound for $m[s, u(z, \bar{z})]$ in terms of $n[s, (u(z, \bar{z}) - \alpha_\nu)^{-1}]$. In this section, we shall apply a modification of the second Fundamental Theorem of Nevanlinna to obtain an upper bound for $m[s, u(z, \bar{z})]$ which will depend largely on $n[s, (u(z, \bar{z}) - \alpha_\nu)^{-1}]$ for q values of α_ν. The modification of the Second Fundamental Theorem will be proved in the Appendix, because the argument involved is not inherently useful in our considerations. The theorem may be stated as follows:

THEOREM 1. *If $f(z) = c_0 + c_k z^k + \cdots$ is an entire function of z, c_0 and c_k not zero, and if $\alpha_1, \alpha_2, \cdots, \alpha_q$ are distinct finite complex numbers, and $\bar{n}[r_1, (f(z) - \alpha_\nu)^{-1}] = 0$, then for $r > r_1$ and $\rho > r$*

$$(3.1)\quad (q-1)m[r, f(z)] \leq [\log (r/r_1)][\sum_{\nu=1}^{q} \bar{n}[r, (f - \alpha_\nu)^{-1}]] - N_1(r) + S_1(r)$$

where (i) $N_1(r) \geq 0$ *for* $r \geq 1$

$$(3.2)\quad \text{(ii)}\quad S_1(r) = 56 + [\log^+ | (1/kc_k) | + 6q \log^+ R_0] + (q+4) \log^+ R + 2q \log (2q/\delta) + q \log 2 + 4 \log^+ (1/r) + 8 \log^+ \rho + 6 \log^+ (1/\rho - r) + [8 \log^+ m(\rho, f) + \sum_{\nu=1}^{q} \log | c_0(r) - \alpha_\nu |]$$

$$\text{(iii)}\quad \delta = \min [| \alpha_n - \alpha_k |, 1] \qquad (n \neq k)$$
$$R = \max | \alpha_\nu |$$
$$R_0 = \max \{|(1/c_0)|, |1/(c_0 - \alpha_\nu)|\}$$

and

THEOREM 1. (a): *If $f(z) = \sum_{n=\lambda}^{\infty} c_n z^n$ is an entire function of z, $\gamma > 0$, and $c_\lambda \neq 0$ and if $\alpha_1, \alpha_2, \cdots, \alpha_q$ are distinct finite complex numbers, and $\bar{n}[r_1, (f(z) - \alpha_\nu)^{-1}] = 0$; then for $r > r_1$ and $\rho > r$, (3.1) holds, where*

(i) $N_1(r) \geq 0$ *for* $r \geq 1$

$$(3.2a)\quad \text{(ii)}\quad S_1(r) = 66 + [\log^+ |(1/\lambda c_\lambda)| + 6q \log^+ R_0] + (4 + 2q) \log^+ R + 2q \log (2q/\delta) + q \log 2 + 9 \log^+ (1/r) + 8 \log^+ \rho + 6 \log^+ (1/\rho - r) + [8 \log^+ m(\rho, f)] + 5 \log^+ | \lambda |$$

(iii) *δ, R are defined as in Theorem 1:*

$$R_0 = \max \{|(1/c_\lambda)|, |(1/\alpha_\nu)|\}.$$

It is of great importance that all terms in $S_1(r)$ which are not in the brackets are independent of the particular function $f(z)$. It is also note-

worthy that if $\alpha_\nu = 0$ in Theorem 1a or $\alpha_\nu = c_0$ in Theorem 1, then R_0 becomes infinite and the inequality (3.2) is useless. This difficulty may be overcome but to do so will be of no value to us because as $c_0 \to \alpha_\nu$, $R_0 \to \infty$ and in our considerations c_0 will not be fixed.

Now, $v_s(z) = \sum_{m=\lambda}^{\infty} c_m(s) z^m$ where $c_m(s)$ is an entire function of s and $c_\lambda(s)$ is the first coefficient which does not vanish identically. If $\lambda = 0$, then suppose that there is a k such that $c_k(s)$ is the next coefficient which does not vanish identically. This is equivalent to assuming that $v_s(z) = c_0(s) + c_k(s) z^k + \cdots$ is not a constant for each fixed s. Because an entire function which is not identically zero can vanish only on an enumerable set of points, one of the following two alternatives holds; except for an enumerable set of values of s along the positive real axis either:

Case 1. $v_s(z) = c_0(s) + c_k(s) z^k + \cdots$ where $c_0(s) \not\equiv 0$ and $c_k(s) \not\equiv 0$

or Case 1a. $v_s(z) = c_\lambda(s) z^\lambda + \cdots$ where $c_\lambda(s) \not\equiv 0$ and $\lambda > 0$.

Thus we may apply Theorem 1 to $f(z) = v_s(z)$ where s is not an element of the above mentioned enumerable set. We obtain

$$(3.3) \quad (q-1) m[r, v_s(z)] \leq [\log(r/r_1)] \sum_{i=1}^{q} \tilde{n}[r_1, (v_s(z) - \alpha_\nu)^{-1}] + S(r, s)$$

where $r_1 = r_1(s)$, $r > r_1(s)$ and $S(r, s)$ is $S_1(r) - N_1(r)$ corresponding to $f(z) = v_s(z)$.

It was pointed out that as $c_0(s) \to \alpha_\nu$, $R_0 \to \infty$, R_0 is a function of s and it is important for the sake of our inequality to have R_0 bounded. It is likewise important for us to have $|1/c_\lambda|$ or $|1/c_k|$ bounded depending on the case under consideration. On the other hand R and δ are independent of s. Our method of attack will be to find that set of values of s for which R_0 and $|(1/c_\lambda)|$ or $|(1/c_k)|$ are bounded. For this purpose, let us define the following sets of values of s along the positive real axis.

$\mathfrak{s}_1(b, c)$ = the set of points for which $|v_s(0) - b| \geq c$ i.e. $|c_0(s) - b| \geq c$

$\mathfrak{s}_2(b, c)$ = " " " " " " $|v_s(0) - b| \leq c$ " $|c_0(s) - b| \leq c$

$\mathfrak{s}_3$ = " " " " " " $c_0(s) \not\equiv 0$ and $c_k(s) \not\equiv 0$ [in case 1]

$\mathfrak{s}_3$ = " " " " " " $c_\lambda(s) \not\equiv 0$ [in case 1a]

$\mathfrak{s}_4(b, r_1)$ = " " " " " " $v_s(z) \neq b$ for $|z| \leq r_1$

$\mathfrak{s}_5(0, c)$ = " " " " " " $|c_k(s)| \geq c$ [in case 1]

$\mathfrak{s}_6(0, c)$ = " " " " " " $|c_\lambda(s)| \geq c$ [in case 1a]

From these definitions it follows that if $c_0(s) \not\equiv 0$, and $s \in \mathfrak{s}_1(\alpha_\nu, c)$

$\cap \hat{S}_1(0, c)$ for $\nu = 1, 2, \cdots q$, then $R_0 \leq 1/c$. Furthermore if $s \varepsilon \hat{S}_5(0, c)$, $|1/c_k| \leq 1/c$. In addition, $N_1(r) \geq 0$ for $r \geq 1$. Thus if $c_0(s) \not\equiv 0$ and $c_k(s) \not\equiv 0$, i. e. $s \varepsilon \hat{S}_3$, we may apply Theorem 1 and the definition of $S(r, s)$ to state:

THEOREM 2: *If*

(i) $c_0(s) \not\equiv 0$, $c_k(s) \not\equiv 0$

(ii) $s \varepsilon \hat{S}_1(0, c) \cap \hat{S}_1(\alpha_\nu c) \cap \hat{S}_5(0, c) \cap \hat{S}_3 \qquad \nu = 1, 2, \cdots q$

(iii) $r \geq 1$

(iv) $\rho > r$

Then there is a constant $c_1[k, c, q, \alpha_\nu]$ *such that*

$$S(r, s) \leq c_1[k, c, q, \alpha_\nu] + 4 \log^+ (1/r) + 6 \log^+ (1/\rho - r) + 8 \log^+ \rho + 8 \log^+ m[\rho, v_s(z)] + \sum_{\nu=1}^{q} \log |c_0(s) - \alpha_\nu|. \tag{3.4}$$

In addition, we have by a similar argument

THEOREM 2a: *If*

(i) $c_\lambda(s) \not\equiv 0$. $\lambda > 0$

(ii) $s \varepsilon \hat{S}_1(\alpha_\nu, c) \cap \hat{S}_6(0, c) \cap \bar{\hat{S}}_3$

(iii) $r \geq 1$

(iv) $\rho > r$

Then there is a constant $c_2[\lambda, c, q, \alpha_\nu]$ *such that*

$$S(r, s) \leq c_2[\lambda, c, q, \alpha_\nu] + 9 \log^+(1/r) + 6 \log^+(1/\rho - r) + 8 \log^+ \rho + 8 \log^+ m[\rho, v_s(z)]. \tag{3.4a}$$

Now, we have from (2.10)

$$m[s, u(z, \bar{z})] = m[s, v_s(z)]$$

and furthermore if there are no α-points of $u(z, \bar{z})$ on $|z| = s$

$$n[s, (u(z, \bar{z}) - \alpha)^{-1}] = n[s, (v_s(z) - \alpha)^{-1}] = \bar{n}[s, (v_s(z) - \alpha)^{-1}].$$

In addition, we have if $s \varepsilon \hat{S}_4(\alpha_\nu, \bar{r})$, $v_s(z) \not\equiv \alpha_\nu$ for $|z| \leq \bar{r}$. Hence $\bar{n}[\bar{r}, (v_s(z) - \alpha_\nu)^{-1}] = 0$. Thus if $s \varepsilon \hat{S}_4(\alpha_\nu, \bar{r})$ $\nu = 1, 2, \cdots q$, then $r_1(s) \geq \bar{r}$. We may now apply (3.3) and Theorems 2 and 2a to the case where $r = s$, whence we obtain:

THEOREM 3: *If*

(i) $c_0(s) \not\equiv 0,\ c_k(s) \not\equiv 0$

(ii) $s \in \hat{S}_1(0, c) \cap \hat{S}_1(\alpha_\nu, c) \cap \hat{S}_5(0, c) \cap \hat{S}_3 \cap \hat{S}_4(\alpha_\nu, \tilde{r}), \quad \nu = 1, 2, \cdots q$

(iii) $s \geq 1$, *and* $s > \tilde{r}$

(iv) *There are no* α_ν*-points of* $u(z, \bar{z})$ *on* $|z| = s$

(v) $\rho > s$

Then

$$(3.5)\quad (q-1)m[s, u(z, \bar{z})] \leq [\log s/\tilde{r}][\sum_{\nu=1}^{q} n[s, (u-\alpha_\nu)^{-1}] + 8\log^+ \rho + 6\log^+ (1/\rho - s) + 8\log^+ m[\rho, v_s(z)] + c_1[k, c, q, \alpha_\nu] + \sum_{\nu=1}^{q} \log |c_0(s) - \alpha_\nu|$$

and we also have

THEOREM 3a: *If*

(i) $c_\lambda(s) \not\equiv 0, \qquad \lambda > 0$

(ii) $s \in \hat{S}_1(\alpha_\nu, c) \cap \hat{S}_6(0, c) \cap \hat{\bar{S}}_3 \cap \hat{S}_4(\alpha_\nu, \tilde{r}) \qquad \nu = 1, 2, \cdots q$

(iii), (iv), (v) *same as in Theorem 3*

Then

$$(3.5a)\quad (q-1)m[s, u(z, \bar{z})] \leq [\log (s/\tilde{r})][\sum_{\nu=1}^{q} n[s, (u-\alpha_\nu)^{-1}]] + 8\log^+ \rho + 6\log^+ (1/\rho - s) + 8\log^+ m[\rho, v_s(z)] + c_2[\lambda, c, q, \alpha_\nu].$$

The inequalities (3.5) and (3.5a) are the inequalities we desired. It is to be noted that the only restriction on ρ is that $\rho > s$. A special case of these inequalities which is of great importance is the case where $\rho = \mu s$, μ constant and greater than 1. Then the term $\log^+ 1/\rho - s$ is bounded and may be incorporated into c_1 or c_2. In addition $\log^+ \rho \leq \log^+ \mu + \log^+ s$ and $\log^+ \mu$ may also be incorporated into c_1 or c_2.

It is also noteworthy that the above inequalities are quite general in the sense that they may apply to *any* function $u(z, \bar{z})$ which takes on the values of $v_s(z)$ for $|z| = s$ where $v_s(z)$ is entire in z and s. These inequalities can now be adapted to the study of classes of solutions of *particular*

partial differential equations by translating the hypotheses of Theorems 3 and 3a into hypotheses concerning the nature of $E^{(1)}(z, s, t)$.

Before we prove such a theorem involving $E^{(1)}(z, s, t)$, let us consider the following example.

Example.

$\Delta u - u = 0$; Here $E(z, \bar{z}, t) = e^{t(z\bar{z})^{\frac{1}{2}}}$ (see [2])

Hence $E^{(1)}(z, s, t) = e^{st}$ and,

$$v_s(z) = \int_{-1}^{1} e^{st} \frac{f[(z/2)(1-t^2)]}{(1-t^2)^{\frac{1}{2}}}\, dt.$$

Suppose that

$f(z) = a_0 + a_k z^k + \cdots$ is entire, $a_0 \neq 0$, $a_k \neq 0$.

Then there exists an $\tilde{r}$ such that for $|z| \leq \tilde{r}$

$$f(z) = a_0[1 + \epsilon_1 e^{i\delta_1}] \quad \text{where} \quad |\epsilon_1| < \tfrac{1}{2}.$$

Thus for $|z| \leq \tilde{r}$

$$v_s(z) = a_0\rho_0(s)[1 + \epsilon_2 e^{i\delta_2}] \tag{3.6}$$

where $|\epsilon_2| < \frac{1}{2}$ and $\rho_0(s) = \int_{-1}^{1} e^{st}(1-t^2)^{-\frac{1}{2}}dt$.

Furthermore

$$v_s(z) = c_0(s) + c_k(s)z^k + \cdots \quad \text{where}$$

$$c_0(s) = a_0 \int_{-1}^{1} e^{st}(1-t^2)^{-\frac{1}{2}}dt = a_0\rho_0(s) \tag{3.7}$$

$$c_k(s) = a_k \int_{-1}^{1} (1/2^k)e^{st}(1-t^2)^{k-\frac{1}{2}}dt = a_k\rho_k(s). \tag{3.8}$$

Since $\rho_0(s)$ and $\rho_k(s)$ tend to infinity as $s \to \infty$, (3.6), (3.7) and (3.8) imply that for s large enough

$$s \in \hat{S}_1(0, c) \cap \hat{S}_1(\alpha_\nu, c) \cap \hat{S}_5(0, c) \cap \hat{S}_3 \cap \hat{S}_4(\alpha_\nu, \tilde{r}), \qquad \nu = 1, 2, \cdots, q.$$

Now suppose that

$$\operatorname*{Max}_{|z|=\mu s} |v_s(z)| = O[e^{s^{\rho+\epsilon}}] \qquad \text{for any positive } \epsilon.$$

Then $\log^+ m[\mu s, v_s(z)] \leq \log^+ \log^+ [e^{s^{\rho+\epsilon}}] \leq (\nu + \epsilon) \log^+ s$.

Furthermore $\log |c_0(s) - \alpha_\nu| \leq c + s$.

Thus, if we apply Theorem 3, (3.5) gives

$$(3.9)\qquad (q-1)m[s,u(z,\bar{z})] \leq [\log s/\tilde{r}][\sum_{\nu=1}^{q} n[s,(u-\alpha_\nu)^{-1}]] + qs + 8(1+\nu+\epsilon)\log s$$

for s large enough and such that $u(z,\bar{z}) \neq \alpha_\nu$ on $|z| = s$. Hence if

$$\sum_{1}^{q} n[s,(u-\alpha_\nu)^{-1}] = O[s^{\delta'+\epsilon}] \qquad \text{for any positive } \epsilon, \text{ and } \delta = \max[\delta', 1],$$

$$(3.10)\qquad m[s,u(z,\bar{z})] = O[s^{\delta'+\epsilon}]$$

for any positive ϵ and for the set of positive s for which $u(z,\bar{z}) \neq \alpha_\nu$. If this set of s is everywhere dense as in the case where all α_ν points of $u(z,\bar{z})$ are isolated points, then (3.10) holds for the entire set of positive s by continuity.

(3.10) shows that $\sum_{\nu=1}^{q} n[s,(u-\alpha_\nu)^{-1}]$ determines an upper bound for $m[s,u(z,\bar{z})]$ unless $m[s,u(z,\bar{z})] = O[s^{1+\epsilon}]$.

The result of this example can be generalized to prove the following

THEOREM 4. *If*

(i) *There exists an $r_2, \tilde{s}, c$, such that for $|z| \leq r_2$, $s > \tilde{s}$*

$$E^{(1)}(z,s,t) = \tilde{E}_0(s,t)[1+\epsilon_1 e^{i\delta_1}] \quad \textit{where} \quad |\epsilon_1| \leq c < 1$$

and $\tilde{E}_0(s,t) = E^{(1)}(0,s,t)$ is real and bounded from below for $s > 0$

(ii) $$E^{(1)}(z,s,t)f[(z/2)(1-t^2)] = a_0\tilde{E}_0(s,t) + z^k\alpha_k(s,t) + \cdots$$
$$a_0 \neq 0,\ \alpha_k(s,t) \not\equiv 0$$

(iii) $$\rho_0(s) = \int_{-1}^{1} \tilde{E}_0(s,t)(dt/(1-t^2)^{1/2}) \to \infty \quad as \quad s \to \infty$$

$$\tilde{\rho}_k(s) = \left| \int_{-1}^{1} \alpha_k(s,t)(1-t^2)^{-\frac{1}{2}}dt \right| \to \infty \quad as \quad s \to \infty$$

(iv) $$h(r) = \max_{|z|=r} |f(z)|$$

$$\rho(s) = \max_{|z|=2s} |E^{(1)}(z,s,t)|$$

Then, there is an $\tilde{r} > 0$ such that

$$(3.11)\qquad (q-1)m[s,u(z,\bar{z})] \leq [\log s/\tilde{r}][\sum_{1}^{q} n[s,(u-\alpha_\nu)^{-1}]] + 8\log^+\log^+[\rho(s)h(s)] + q\log\rho_0(s) + 8\log s + c$$

for s large enough and such that

$$u(z,\bar{z}) \neq \alpha_\nu \text{ on } |z| = s.$$

Proof. $f(z) = \sum_0^\infty a_n z^n$, $a_0 \neq 0$. Thus there is an r_3 such that $f(z) = a_0(1 + \epsilon_3 e^{i\delta_3})$ where $|\epsilon_3| < (1-c)/16$ for $|z| \leq r_3$. Now $\tilde{r} = \min(r_2, r_3)$. Then

$$v_s(z) = \int_{-1}^{1} \tilde{E}_0(s,t) a_0 [1 + \epsilon_4 e^{i\delta_4}] (dt/(1-t^2)^{1/2})$$

where $|\epsilon_4| < 1 - (14/16)(1-c) < 1$ for $|z| \leq \tilde{r}$, $s > \tilde{s}$. Now, since $\tilde{E}_0(s,t)$ is bounded from below for $s > 0$, there exists an M_1 such that

$$E_0(s,t) = -M_1 + H(s,t) \text{ where } H(s,t) > 0 \text{ for } s > 0.$$

Using the fact that $\rho_0(s) \to \infty$, we have

$$\gamma_0(s) = \int_{-1}^{1} (H(s,t)/(1-t^2)^{1/2})\, dt \qquad \text{as } s \to \infty.$$

Indeed $\gamma_0(s)/\rho_0(s) \to 1$ as $s \to \infty$.

Now $v_s(z) = -a_0 M_1 \int_{-1}^{1} \frac{(1 + \epsilon_4 e^{i\delta_4})}{(1-t^2)^{1/2}}\, dt + \int_{-1}^{1} \frac{H(s,t) a_0 (1 + \epsilon_4 e^{i\delta_4})}{(1-t^2)^{1/2}}\, dt$

$= a_0 N_1 + a_0 \gamma(s) [1 + \epsilon_5 e^{i\delta_5}]$

where $|N_1| \leq 2M_1 \int_{-1}^{1} (dt/(1+t^2)^{1/2})$

and $|\epsilon_5| \leq 1 - (14(1-c)/16)$ for s large enough and $|z| \leq \tilde{r}$.

Thus since $\gamma_0(s) \to \infty$

$v_s(z) = a_0 \gamma_0(s) [1 + \epsilon_6 e^{i\delta_6}]$ where $|\epsilon_6| \leq 1 - (12/16)(1-c)$ for $|z| \leq \tilde{r}$, s large enough or

(3.12) $v_s(z) = a_0 \rho_0(s) [1 + \epsilon_7 e^{i\delta_7}]$ where $|\epsilon_7| \leq 1 - (10/16)(1-c)$ for $|z| \leq \tilde{r}$, s large enough.

Furthermore

(3.13) $|c_0(s)| = |a_0 \rho_0(s)| \to \infty$

(3.14) $|c_k(s)| = |\tilde{\rho}_k(s)| \to \infty$.

Thus for s large enough $s \in \hat{S}_1(0,c) \cap \hat{S}_1(\alpha_\nu, c) \cap \hat{S}_5(0,c) \cap \hat{S}_3 \cap \hat{S}_4(\alpha_\nu, \tilde{r})$. And for $|z| = 2s$

$$|v_s(z)| = \left| \int_{-1}^{1} E^{(1)}(z,s,t) f[(z/2)(1-t^2)] (dt/(1-t^2)^{1/2}) \right|$$

$$\leq \rho(s) h(s) \int_{-1}^{1} dt/(1-t^2)^{1/2}.$$

Hence

$$\log^+ m[2s, v_s(z)] \leq \log^+ \log^+ \rho(s) h(s) + \log \log \pi$$

and

$$\log | c_0(s) - \alpha_\nu | \leq c + \log | c_0(s) | \leq c^1 + \log \rho_0(s) .$$

Applying Theorem 3 where $\rho = 2s$, we have for all s large enough and such that $u(z, \bar{z}) \neq \alpha_\nu$ on $| z | = s$, (3.11) which is what we wished to prove. If we know that $\underset{|z|=2s}{\text{Max}} | v_s(z) | = w(s)$, it is evident that in (3.11) $\log^+ \log^+ h(s)\rho(s)$ may be replaced by $\log^+ \log^+ w(s)$.

In Theorem 4 the inequality for $m[s, u(z, \bar{z})]$ is stated with reference to certain properties of $E^{(1)}(z, s, t)$. In **5**, certain classes of partial differential equations with known generating functions will be investigated with reference to Theorem 4.

4. An Analogue of the Picard Theorem. The following statement is a simple corollary of Rouché's Theorem: If $f(z)$ is analytic in the circle $| z | \leq r$, $f(0) = 0$ and $| f(z) | \geq | \alpha |$ on $| z | = r$ then $f(z)$ assumes the value α at least once in the circle $| z | < r$. This corollary provides a method of proving an analogue of the Picard Theorem for certain sets [7] of complex solutions of elliptic equations. The method of attack will first be illustrated by an example.

Example. $\Delta u - u = 0$ $\quad E^{(1)}(z, s, t) = e^{st}$. Let $f(z) = \sum_{n=\lambda}^{\infty} a_n z^n$ where $a_\lambda \neq 0$, $\lambda > 0$. Then, there exists an $\bar{r}$ such that for $| z | \leq \bar{r}$, $f(z) = a_\lambda z^\lambda [1 + \epsilon_1 e^{i\delta_1}]$, $| \epsilon_1 | < \frac{1}{2}$, so that for $| z | \leq \bar{r}$

$$v_s(z) = (1/2^\lambda) \int_{-1}^{1} e^{st} a_\lambda z^\lambda [1 + \epsilon_1 e^{i\delta_1}] (1 - t^2)^{\lambda - \frac{1}{2}} dt = a_\lambda z^\lambda \rho_\lambda(s) [1 + \epsilon_2 e^{i\delta_2}]$$

where $| \epsilon_2 | < \frac{1}{2}$ and where $\rho_\lambda(s) = (1/2^\lambda) \int_{-1}^{1} e^{st} (1 - s^2)^{\lambda - \frac{1}{2}} dt \to \infty$ as $s \to$

Thus for $| z | = \bar{r}$

$$| v_s(z) | \geq \tfrac{1}{2} | a_\lambda | \bar{r}^\lambda \rho_\lambda(s) \to \infty . \tag{4.1}$$

Consider any finite complex number α and suppose that $u(z, \bar{z})$ does not attain the value α. Then $n[s, (u - \alpha)^{-1}]$ is defined and

$$n[s, (u(z, \bar{z}) - \alpha)^{-1}] = \bar{n}[s, (v_s(z) - \alpha)^{-1}] = 0 .$$

Now $v_s(0) = 0$ and for $s > \bar{r}$ and s large enough and $| z | = \bar{r}$, $| v_s(z) | > | a |$. Thus $v_s(z)$ asumes the value α somewhere in $| z | \leq \bar{r}$. Hence

[7] These sets are subsets of certain classes of complex solutions.

$$n[s,(u(z,\bar{z})-\alpha)^{-1}] = \tilde{n}[s,(v_s(z)-\alpha)^{-1}] \geq \tilde{n}[\tilde{r},(v_s(z)-\alpha)^{-1}] \geq 1$$

and our original asumption that α is not attained by $u(z,\bar{z})$ is contradicted. This means that $u(z,\bar{z})$ assumes all finite values if the associate function $f(z)$ vanishes at the origin.

The above argument suggests the following theorem.

THEOREM 5. *If*

(i) *There exists an* r_2, $\tilde{s}$, c, *such that for* $|z| \leq r_2$, $s > \tilde{s}$

$$E^{(1)}(z,s,t) = \tilde{E}_0(s,t)[1+\epsilon_1 e^{i\delta_1}] \quad \textit{where} \quad |\epsilon_1| \leq c < 1$$

and $\tilde{E}_0(s,t) = E^{(1)}(0,s,t)$ *is real and bounded from below for* $s > 0$

(ii) $$f(z) = \sum_{n=\lambda}^{\infty} a_n z^n \qquad a_\lambda \neq 0, \quad \lambda > 0$$

(iii) $$\rho_\lambda(s) = (1/2^\lambda) \int_{-1}^{1} \tilde{E}_0(s,t)(1-t^2)^{\lambda-\frac{1}{2}}dt \to \infty \qquad \textit{as } s \to \infty.$$

Then all finite values are assumed by $u(z,\bar{z})$.

Proof. Since $f(z) = \sum_{n=\lambda}^{\infty} a_n z^n$ $a_\lambda \neq 0$, $\lambda > 0$, there exists an r_3 such that for $|z| \leq r_3$

$$f(z) = z^\lambda a_\lambda[1+\epsilon_3 e^{i\delta_3}] \qquad \text{where } |\epsilon_3| \leq (1/16)(1-c).$$

Following through the argument used to get (3.12) in the proof of Theorem 4, we obtain

$$(4.2) \qquad v_s(z) = z^\lambda a_\lambda[1+\epsilon_7 e^{i\delta_7}]\left[\int_{-1}^{1} (1/2^\lambda)\tilde{E}_0(s,t)(1-t^2)^{\lambda-\frac{1}{2}}dt\right]$$
$$= z^\lambda a_\lambda \rho_\lambda(s)[1+\epsilon_7 e^{i\delta_7}]$$

where $|\epsilon_7| < 1-(10/16)(1-c)$ for $|z| \leq \tilde{r} = \min[r_2, r_3]$, s large enough. Thus we have

(i) $v_s(0) = 0$

(ii) $|v_s(z)| \geq \tilde{r}^\lambda |a_\lambda| \rho_\lambda(s)(10/16)(1-c)$ for s large enough and $|z| = \tilde{r}$. Now given any value α, if $u(z,\bar{z})$ does not attain the value α, then $n[s,(u-\alpha)^{-1}]$ is defined and

$$n[s,(u(z,\bar{z})-\alpha)^{-1}] = \tilde{n}[s,(v_s(z)-\alpha)^{-1}] = 0.$$

But for s large enough, $s > \tilde{r}$ and $|z| = \tilde{r}$, $|v_s(z)| > |\alpha|$. Thus $v_s(z)$ assumes the value α in $|z| \leq \tilde{r}$. Hence

$$n[s,(u(z,\bar{z})-\alpha)^{-1}] = \tilde{n}[s,(v_s(z)-\alpha)^{-1}] \geq \tilde{n}[\tilde{r},(v_s(z)-\alpha)^{-1}] \geq 1$$

and this contradiction proves our theorem.

In Theorems 4 and 5 the conditions on $E^{(1)}(z,s,t)$ may be relaxed somewhat by the following consideration. Suppose that $E^{(2)}(z,s,t)$ satisfies the conditions of these theorems and that

$$E^{(1)}(z,s,t) = g[(z/2)(1-t^2)]E^{(2)}(z,s,t)$$

where $g(\zeta)$ is an entire function of ζ. Then it is evident that the class of entire functions generated by $E^{(1)}$ is included in the class generated by $E^{(2)}$. Thus the theorems hold for the class generated by $E^{(1)}$ because they hold for the class generated by $E^{(2)}$.

It is worth mentioning that Theorem 3 suggests another attack on the analogue of the Picard Theorem. If we suppose that there are two finite complex numbers α_1 and α_2 which are not attained by $u(z,\bar{z})$, we have

$$n[s,(u(z,\bar{z})-\alpha_1)^{-1}] = n[s,(u(z,\bar{z})-\alpha_2)^{-1}] = 0.$$

Then Theorem 3 states that for a certain set of positive values of s for which conditions (ii), (iii) and (iv) are satisfied.

$$(4.3) \qquad m[s,u(z,\bar{z})] < 8\log^+\rho + 8\log^+ m[\rho,v_s(z)] + 6\log^+(1/\rho - s) + \sum_{\nu=1}^{2}\log|c_0(s)-\alpha_\nu| + c.$$

If we could prove that there is a set of values of s for which (ii), (iii) and (iv) are satisfied and which has elements tending to ∞, and there exists an $h(s)\to\infty$ such that

$$(4.4) \qquad m[s,u(z,\bar{z})] \geq 8\log\rho + 8\log^+ m[\rho,v_s(z)] + 6\log^+(1/\rho - s) + \sum_{\nu=1}^{2}\log|c_0(s)-\alpha_\nu| + h(s)$$

the resulting contradiction would imply that $u(z,\bar{z})$ assumes one of the two values α_1 and α_2.

5. Examples. Thus far, all theorems which refer to particular partial differential equations refer to them in terms of $E^{(1)}(z,s,t)$ only. It would be of great value to be able to state theorems in terms of the coefficients of the partial differential equations.

Bergman (See [2, § 3] and see also [7]) has given a method of determining E functions corresponding to equations with certain coefficients. Suppose that the equation has the form

$$(5.1) \qquad (\Delta u/4) + A(x,y)u_x + B(x,y)u_y + C(x,y)u = 0$$

or

$$(5.1') \qquad u_{z\bar{z}} + a(z,\bar{z})u_z + \overline{a(z,\bar{z})}u_{\bar{z}} + c(z,\bar{z})u = 0$$

where we define

$$f_z = \tfrac{1}{2}[f_x - if_y] \qquad f_{\bar{z}} = \tfrac{1}{2}[f_x + if_y]\ ^{8}.$$

Then if (i) $a_{\bar{z}} = \bar{a}_z$

(ii) $F_z = 0$ where $F = -a_z - |a|^2 + c(z,\bar{z})$

it follows that

$E(z,\bar{z},t) = \exp P(z,\bar{z},t)$ is a generating function of complex solutions of (5.1)
where

$$P(z,\bar{z},t) = -\int_0^{\bar{z}} a\,d\bar{z} + z^{\frac{1}{2}}[-4\int_0^{\bar{z}} F d\bar{z}]^{\frac{1}{2}}t.$$

We can now apply our theorems to some of these equations. Consider, for example, the case where

(i) $a(z,\bar{z}) = z\sum_0^{\sigma} c_n(z\bar{z})^n$, c_n real. Hence a_z is also real.

(ii) $c(z,\bar{z}) = a_z + |a|^2 - \gamma_0^2$, γ_0 real. Hence c is also real.

Then $a_z = \bar{a}_{\bar{z}}$, $F = \gamma_0^2$ and $F_z = 0$ whence

$$E(z,\bar{z},t) = \exp P(z,\bar{z},t) = \exp\left[-\sum_0^{\sigma}(c_n/n+1)(z\bar{z})^{n+1} + 2(z\bar{z})^{\frac{1}{2}}\gamma_0 t\right]$$

$$E^{(1)}(z,s,t) = \exp\left[-\sum_0^{\sigma}(c_n/n+1)s^{2(n+1)} + 2s\gamma_0 t\right]$$

and hence $E^{(1)}(z,s,t) \geq 0$.

We may now consider the case where

(iii) $a(z,\bar{z}) \not\equiv 0, \qquad c_\sigma < 0$

then

$E^{(1)}(z,s,t)$ has the magnitude of $\exp\left[-(c_\sigma/\sigma+1)s^{2(\sigma+1)}\right]$

$\rho_n(s) \geqq c\exp\left[-(\mu c_\sigma/(\sigma+1))s^{2(\sigma+1)}\right]$ for $0 < \mu < 1$, μ fixed.

Applying Theorem 4, we have, if $f(z) = a_0 + a_k z^k + \cdots \qquad a_0 \neq 0,\ a_k \neq 0,$

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[8] This notation has the advantage that quantities like $u = z^r\bar{z}^s$ may be differentiated formally i.e. $u_z = rz^{r-1}\bar{z}^s$, $u_{\bar{z}} = sz^r\bar{z}^{s-1}$.

$$(5.2)\quad (q-1)m[s,u(z,\bar{z})] \leq [\log s/\bar{r}][\sum_1^q n[s,(u-\alpha_\nu)^{-1}]]$$
$$+ 8\log^+\log^+ h(s) + (\epsilon + |qc_\sigma|)/(\sigma+1)s^{2(\sigma+1)}$$

for s large enough and such that $u(z,\bar{z}) \neq \alpha_\nu$ on $|z| = s$. Applying Theorem 5, we have, if $f(z) = a_\lambda z^\lambda + \cdots \quad a_\lambda \neq 0,\ \lambda > 0$. Then $u(z,\bar{z})$ assumes all finite values.

An example of one of the equations considered above is

$$u_{z\bar{z}} + (-z)u_{\bar{z}} + (-\bar{z})u_z + [-1 + |z|^2 - 1]u = 0$$

i. e.

$$(\Delta u/4) - xu_x - yu_y + (x^2 + y^2 - 2)u = 0$$

and

$$E^{(1)}(z,s,t) = \exp[s^2 + 2st].$$

For some of the equations above we can get other classes of solutions. If the following conditions are satisfied

(a) $F_z = 0 \qquad F = -a_z - |a|^2 + c(z,\bar{z})$

(b) $F = (a_z - \bar{a}_{\bar{z}}/2)$

then we have for E (See [2, § 3])

$E(z,\bar{z},t) = \exp[P(z,\bar{z},t)]$ where

$$P(z,\bar{z},t) = -\int_0^{\bar{z}} ad\bar{z} + z[c + 2\int_0^{\bar{z}} Fd\bar{z}]t^2,\ c \text{ an arbitrary constant.}$$

It can be shown without much trouble that our theorems apply to these new classes of solution if

$$a(z,\bar{z}) = z\sum_{n=0}^{\sigma} c_n(z\bar{z})^n,\ c_n \text{ real},\ c(z,\bar{z}) = a_z + |a|^2.$$

6. Pseudo-Simple Functions. A classical theorem in the theory of analytic functions of a complex variable states that if $f(z) = z + c_2z^2 + c_3z^3 + \cdots$ is simple[9] in the circle $|z| < 1$, then (See [6])

$$|c_m| \leq k(m) \qquad m = 2, 3, \cdots$$

where

$$k(2) = 2, \qquad k(3) = 3, \qquad k(4) = 4.2848, \qquad k(5) = 5.9158 \text{ etc.}$$

There exists a conjecture that if $f(z)$ is simple in $|z| < 1$, $|c_m| \leq m$.

[9] A function $f(z)$ is said to be simple in a domain if $f(z)$ is analytic and simple valued and if for every two distinct points z_1 and z_2 of the domain $f(z_1) \neq f(z_2)$.

If $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is simple in a domain it can have no points of multiplicity greater than one. Thus if $f(z)$ is simple in $|z| < \rho$, then

(6.1) $\quad \tilde{n}[r,(f-\alpha)^{-1}] = 0$ or 1 for all α and all $r < \rho$.

On the other hand if

(6.2) $\quad \tilde{n}[r,(f-\alpha)^{-1}] = 0$ or 1 for all α

then $f(z)$ is simple in $|z| < r$ and thus

(6.3) $\quad |a_m| \leq k(m) r^{m-1} |a_1| \qquad m = 2, 3, 4, \cdots.$

We can now set up an analogue of the concept of simple functions and establish a theorem corresponding to the above for functions $u(z,\bar{z}) \in \mathcal{B}(E)$ where E is a generating function of the type we have been considering.

DEFINITION. *Suppose that $u(z,\bar{z}) = v_s(z)$ for $|z| = s$ where $v_s(z)$ is an entire function of the two complex variables z and s. Then $u(z,\bar{z})$ is said to be pseudo-simple in $|z| < \rho$ if*

(6.4) $\quad \tilde{n}[s,(v_s(z)-\alpha)^{-1}] = 0$ *or* 1 *for all* α *and all* s *such that* $0 < s < \rho$.

It is to be noted that if $u(z,\bar{z})$ has no α-points on $|z| = s$, then

$$n[s,(u(z,\bar{z})-\alpha)^{-1}] = \tilde{n}[s,(v_s(z)-\alpha)^{-1}] = 0 \text{ or } 1.$$

We have defined pseudo-simple in terms of $v_s(z)$ instead of $u(z,\bar{z})$ to avoid complications which occur when $u(z,\bar{z})$ has α-points on $|z| = s$. It would also have been possible to define this concept directly in terms of $u(z,\bar{z})$ and for a more general set of functions $u(z,\bar{z})$.

If the function $u(z,\bar{z})$ is pseudo-simple in $|z| < \rho$, it does not necessarily follow that the inverse function of $u(z,\bar{z})$ is single valued as is the case for simple functions. For example consider the case of

$$u(z,\bar{z}) = (zJ_1(|z|))/|z|) = (z/2)[1-(1/8)(z\bar{z}) + \cdots]$$

Then $v_s(z) = zJ_1(s)/s$. If $J_1(s) \neq 0$, $\tilde{n}[s,(v_s(z)-\alpha)^{-1}] = 0$ or 1 for each α. If s_1 is the smallest $s > 0$ such that $J_1(s) = 0$ then our function is pseudo simple in $|z| < s_1$.

On the other hand if ϵ is a positive number which is small enough, then $J_1(s)$ assumes the value ϵ twice in the interval $0 < s < s_1$. Hence $u(z,\bar{z})$ assumes ϵ twice in $|z| < s_1$. The reason that $\tilde{n}$ remains less than two is that the index of the second ϵ point is -1 and cancels that of the first.

From our definition we see that if $u(z, \bar{z})$ is pseudo-simple in $|z| < \rho$ and $v_s(z) = \Sigma c_{mn} z^m s^n$ then

$$(6.5) \qquad |c_m(s)| \leq k(m) s^{m-1} |c_1(s)|, \qquad m = 2, 3, \cdots, \quad 0 < s < \rho$$

where $$c_m(s) = \sum_{n=0}^{\infty} c_{mn} s^n .$$

If $u(z, \bar{z}) = \Sigma A_{mn} z^m \bar{z}^n$, $m > n$, then $v_s(z) = \Sigma A_{mn} z^{m-n} s^{2n}$. Thus we have $c_m(s) = \Sigma A_{m+n,n} s^{2n}$. Furthermore, as was pointed out in (2.4), if $u(z, \bar{z}) \in \mathcal{B}(E)$ where $E^{(1)}(z, s, t) = \Sigma E_\mu^{(m)}(t) z^u s^m$ then

$$c_{mn} = \sum_{\nu=0}^{n} \frac{A_{\nu 0} \Gamma(\nu + 1)}{\pi^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2})} \int_{-1}^{1} E_{n-\nu}^{(m)}(t) (1 - t^2)^{\nu - \frac{1}{2}} dt .$$

Thus we have

THEOREM 6. *If $u(z, \bar{z}) = \Sigma A_{mn} z^m \bar{z}^n$ belongs to $\mathcal{B}(E)$ where $E^{(1)}(z, s, t) \Sigma E_\mu^{(m)}(t) z^\mu s^m$ and $u(z, \bar{z})$ is pseudo-simple in $|z| < \rho$, then*

$$(6.5) \qquad |c_m(s)| \leq k(m) s^{m-1} |c_1(s)| \qquad m = 2, 3, \cdots \quad 0 < s < \rho$$

where

$$(6.6) \qquad c_m(s) = \sum_{n=0}^{\infty} A_{m+n,n} s^{2n}$$

and also

$$(6.7) \qquad c_m(s) = \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} \frac{A_{\nu 0} \Gamma(\nu + 1)}{\pi^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2})} \left[\int_{-1}^{1} E_{n-\nu}^{(m)}(t) (1 - t^2)^{\nu - \frac{1}{2}} dt \right] s^n .$$

Theorem 6 establishes as necessary conditions for $u(z, \bar{z}) \in \mathcal{B}(E)$ to be pseudo-simple certain relations expressed as inequalities in terms of the coefficients of the expansion of $u(z, \bar{z})$. When (6.6) is used we consider for each m an inequality in terms of the coefficients $A_{m+n,n}$. When (6.7) is used we obtain for each m an inequality in terms of $\{A_{\nu 0}\}$. In these inequalities the functions $E_m^n(t)$ occur. These functions are independent of the particular function $u(z, \bar{z}) \in \mathcal{B}(E)$ which we are considering. It is noteworthy that instead of (6.7) we can get relations involving $\{A_{\nu r}\}$ for any fixed r.

In the case where $E(z, \bar{z}, t)$ is a generating function of the first kind i. e. $E(z, 0, t) = c_1, E(0, \bar{z}, t) = c_2$ (See [1]), it is possible to relax the condition on $u(z, \bar{z})$. It is sufficient to assume that $u(z, \bar{z})$ is an analytic function of the real variables x and y everywhere in the finite plane. This is enough because this condition implies that the associate function $f(\zeta)$ is entire (See [3, p. 300]) and then $u(z, \bar{z})$ can be expanded in the power series $\Sigma A_{mn} z^m \bar{z}^n$, $m > n$.

Appendix. A Modification of Nevanlinna's Theorem. The second theorem of Nevanlinna was stated in a form which was more convenient for our purpose than the one presented in [4, § 33 and 34]. However our modification can be established by a simple change of Nevanlinna's arguments. We start from ([4], p. 65, (15))

$$(7.1)\qquad m[r,F] > \sum_{\nu=1}^{q} m[r,(f-\alpha_\nu)^{-1}] - q\log(2q/\delta) - \log 3$$

$$F = \sum_{\nu=1}^{q} (1/f-\alpha_\nu)$$

and from [4, p. 66]

$$(7.2)\qquad m[r,F] \leq 2T(r,f) + m(r,(f'/f)) + m(r,\Sigma(f'/f-\alpha_\nu)) - m(r,f) - N_1(r) + \log|(1/kc_k)|$$

where $f(z) = c_0 + c_k z^k + \cdots$ is a meromorphic function and $c_0 \neq 0$, $c_k \neq 0$ and

$$N_1(r) = [2N(r,f) - N(r,f')] + N(r,1/f').$$

Now, if we asume that $f(z)$ is entire, then

$$N(r,f) = N(r,f') = 0$$

$$N_1(r) = N(r,1/f') \geq 0 \text{ for } r \geq 1$$

by the definition of $N(r,f)$ given in [4, p. 6].

We also have $m[r,f] = T[r,f]$. Hence (7.1) and (7.2) together give

$$(7.3)\qquad \sum_{\nu=1}^{q} m[r,(f-\alpha_\nu)^{-1}] \leq q\log(2q/\delta) + \log 3 + m(r,f) + m(r,f'/f) + m(r,\Sigma(f'/f-\alpha_\nu)) - N_1(r) + \log|(1/kc_k)|$$

or

$$(7.3')\qquad \sum_{\nu=1}^{q} m[r,(f-\alpha_\nu)^{-1}] \leq m(r,f) + S_0(r) - N_1(r)$$

where

$$(7.4)\qquad S_0(r) = m(r,\Sigma(f'/f-\alpha_\nu) + m(r,f'/f) + \log|3/kc_k| + q\log 2q/\delta.$$

From [4, p. 67] we have

$$(7.5)\qquad m(r,\Sigma(f'/f-\alpha_\nu)) < 24 + 4\log 3 + 4\log q + 4\log^+\log^+ R + 2\log^+ 1/r + 4\log^+\rho + 3\log^+ 1/\rho - r + 4\log^+ T(\rho,f) + 3\log^+|1/\Phi(0)|$$

where $\log^+|1/\Phi(0)| \leq q\log^+ R_0$.

In addition we have from [4, p. 61]

$$(7.6)\qquad m(r,f'/f) < 24 + 3\log^+ |\,1/c_0\,| + 2\log^+ 1/r + 4\log^+ \rho + 3\log^+ 1/\rho - r + 4\log^+ T(\rho,f)$$

and because $R_0 \geq |\,1/c_0\,|$, we have

$$(7.7)\qquad S_0(r) < 56 + \log^+ |\,1/kc_k\,| + 6q\log^+ R_0 + 4\log^+\log^+ R + 2q\log 2q/\delta + 4\log^+ 1/r + 8\log^+\rho + 6\log^+ 1/\rho - r + 8\log^+ T(\rho,f).$$

From [4, p. 11] we have

$$(7.8)\qquad T[r,(f-a)] - T[r,1/f-a] = \log|\,c'_\lambda\,|$$

if $f(z) - a = c'_\lambda z^\lambda + c_{\lambda+1}z^{\lambda+1} + \cdots$ and $f(z)$ is non-constant.

This together with

$$(7.9)\qquad |\,T[r,f] - T[r,f-a]\,| \leq \log^+|\,a\,| + \log 2$$

gives

$$(7.10)\qquad m[r,(f-a)^{-1}] + N[r,(f-a)^{-1}] + \log|\,c'_\lambda\,| + \log^+|\,a\,| + \log 2 \geq m[r,f]$$

whence (7.3′) becomes

$$(7.11)\qquad (q-1)m[r,f] \leq \sum N[r,(f-\alpha_\nu)^{-1}] - N_1(r) + S_1(r)$$

where

$$(7.12)\qquad S_1(r) \leq S_0(r) + \sum\log^+|\,\alpha_\nu\,| + q\log 2 + \sum\log|\,c_0-\alpha_\nu\,| \text{ if } |\,c_0-\alpha_\nu\,| \neq 0$$

and we obtain our inequality for $S_1(r)$.

$$(3.2)\qquad S_1(r) < 56 + \log^+|\,1/kc_k\,| + 6q\log^+ R_0 + (4+q)\log^+ R + 2q\log 2q/\delta + 4\log^+ 1/r + 8\log^+\rho + 6\log^+ 1/\rho - r + 8\log^+ T(\rho,f) + \sum\log|\,c_0-\alpha_\nu\,| + q\log 2.$$

Now if $f(z) = \sum_\lambda^\infty c_n z^n$ is entire where $\lambda > 0$, these arguments must be revised. In the main we shall make use of the same attack.

Equation 7.1 still holds. However the proof of 7.2 in Nevanlinna involved c_0. We can go through a similar argument with very few changes. From [4, p. 65] we have

$$(7.13)\qquad m[r,F] \leq m[r,1/f] + m[r,f/f'] + m[r,\Sigma f'/(f-\alpha_\nu)]$$

and (7.8) applied when $a=0$ gives

$$(7.14)\qquad m(r,1/f) = T(r,f) - N[r,1/f] + \log|1/c_\lambda|.$$

We can apply (7.8) to the function $g = f'/f = \lambda/z + \cdots$ and

$$(7.15)\qquad m(r,f/f') = N[r,f'/f] - N[r,f/f'] + \log|1/\lambda| + m[r,f'/f]$$

$$(7.16)\qquad m[r,F] \leq 2T(r,f) + m[r,\Sigma f'/(f-\alpha_\nu)] - m(r,f) - N_1(r) + \log|1/\lambda c_\lambda| + m[r,f'/f]$$

where $N_1(r) = N(r,1/f) + N(r,f/f') - N(r,f'/f) \geq 0$ for $r \geq 1$.

Combining this with (7.1) we have equation (7.3′) where

$$(7.17)\qquad S_0(r) = m[r,\Sigma f'/(f-\alpha_\nu)] + m(r,f'/f) + \log|\,|3/\lambda c_\lambda|\, + q\log 2q/\delta.$$

Now we may make use of the footnote in [4, p. 61].

$$(7.18)\qquad m(r,f'/f) < 34 + 5\log^+|\lambda| + 3\log^+|1/c_\lambda| + 7\log^+ 1/r + 4\log^+\rho + 3\log^+(1/\rho - r) + 4\log^+ T(\rho,f).$$

The inequality (7.18) was proved on the assumption that the expansion of f has at least two non-vanishing terms. However if $f(z) = c_\lambda z^\lambda$, $f'/f = \lambda/z$ and thus

$$m[r,f'/f] = \log^+|\lambda/r| \leq \log^+|\lambda| + \log^+ 1/r$$

and the inequality (7.18) holds.

Now, on the assumption that $c_0 \neq \alpha_\nu$, i.e. $\alpha_\nu \neq 0$, inequality (7.5) is derived where

$$R_0 = \max\{|1/c_\lambda|, |1/\alpha_\nu|\}$$

and

$$\Phi(0) = \Pi(c_0 - \alpha_\nu) = \Pi(-\alpha_\nu)$$

and thus

$$\log^+|1/\Phi(0)| \leq q\log^+ R_0 \text{ as before.}$$

Combining (7.5), (7.18), (7.17), we have, for $q \geq 1$, and on making a few simplifications

$$(7.19)\qquad S_0(r) < 66 + \log^+|1/\lambda c_\lambda| + 6q\log^+ R_0 + 4\log^+\log^+ R + 2q\log 2q/\delta + 9\log^+ 1/r + 8\log^+\rho + 6\log^+(1/\rho - r) + 8\log^+ m(\rho,f) + 5\log^+|\lambda|.$$

Now from equation (7.3′) we can go to (7.11) by making use of (7.10). In (7.10) c'_λ stands for the first coefficient of $f(z) - a$. If $a \neq 0$, we have $c'_\lambda = -a$ because there is no constant term in $f(z)$. Thus we have

$$(7.11) \qquad (q-1)\ m[r,f] \leq \Sigma N[r,(f-\alpha_\nu)^{-1}] - N_1(r) + S_1(r)$$

where

$$(7.12) \qquad S_1(r) \leq S_0(r) + \Sigma \log^+ |\alpha_\nu| + q \log 2 + \Sigma \log |\alpha_\nu| \leq S_0(r) + 2q \log^+ R + q \log 2$$

and thus

$$(3.2a) \qquad S_1(r) < 66 + \log^+ |1/\lambda c_\lambda| + 6q \log^+ R_0 + (4+2q) \log^+ R + 2q \log(2q/\delta) + 9 \log^+ (1/r) + 8 \log^+ \rho + 6 \log^+ (1/\rho - r) + 8 \log^+ m(\rho, f) + 5 \log^+ |\lambda| + q \log 2.$$

Finally, we have, if $\tilde{n}[r_1, (f-\alpha_\nu)^{-1}] = 0$, $r > r_1$

$$N[r,(f-\alpha_\nu)^{-1}] = \int_{r_1}^{r} (\tilde{n}[t,(f-\alpha_\nu)^{-1}]/t)\,dt \leq \tilde{n}[r,(f-\alpha_\nu)^{-1}] \log r/r_1$$

whence we obtain our desired result

$$(3.1) \qquad (q-1)\,m[r,f(z)] \leq [\log r/r_1][\sum_{\nu=1}^{q} \tilde{n}[r,(f-\alpha_\nu)^{-1}]] - N_1(r) + S_1(r).$$

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THE INTRODUCTION OF LOCAL CONNECTIVITY BY CHANGE OF TOPOLOGY.*

By GAIL S. YOUNG, JR.

1. Introduction. Suppose that S is a topological space, and that G is a collection of subsets of S. I shall say that a point P of S is a G-limit point of a subset M of S provided that every open neighborhood of P contains an element of G that intersects both P and $M - P$. A set is G-closed if it contains all its G-limit points, and is G-open if its complement is G-closed. These definitions determine a new topology for S—the G-topology—which is actually a "topology" in the sense that if S is a T_0 space originally it is still one in the G-topology. This is shown in Theorem 1. This G-topology is consistent with the original topology in that every G-limit point of a set is a limit point of that set in the original topology, but is weaker[1] in the sense that there are "less" limit points. The motivation of this paper is in the fact that for some choices of G, S is locally connected in the G-topology, though it may not have been so originally. This is the case, for example, if G is the collection of all arcs of S. When this does occur, it is often possible to prove a theorem about S by changing to a G-topology and using properties of locally connected spaces. For example, consider the following theorem, due to Whyburn:[2] A complete metric space S is arcwise connected if and only if each two points of S can be joined by a connected and locally connected subset of S. By taking G to be the collection of all locally connected and connected subsets of S, we find from Theorems 3, 4, and 5 that in the G-topology S is still complete metric and connected, and is now locally connected. Arcwise connectivity follows immediately. In a similar fashion, virtually any theorem on general locally connected spaces can be used to give some sort of theorem on non-locally connected spaces—though usually a theorem of no particular interest.

Rather little is yet known about general locally connected spaces.[3] It seems likely that the concepts of this paper will become more useful when

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[1] Cf. Birkhoff [1]. Numbers in square brackets refer to the bibliography.

[2] Essentially the second theorem on page 756 of [23].

[3] Of items occurring in the bibliography of this paper, [8], [14], [21], and [22] are particularly important in this connection.

more results are available. The notion of a G-topology may possibly be useful when the resulting space is not locally connected, but I have found no example of this.

One application of these concepts seems interesting enough to deserve special mention. That is the following theorem: *Let M be any set that contains no arc, let D be a dendrite, and let H be a collection of arcs such that each point of M is joined to D by some arc of H, and such that $M + D + H^*$ contains only one arc between any two points of M. Then $M + D + H^*$ has the fixed-point property.*[4] Section **4** is devoted to the proof of this result. As a preliminary, study is made in Section **3** of a type of generalized dendrite, with particular emphasis on the fixed-point property.

There is a strong relationship between the concepts of this paper and certain work of Menger, Whyburn, and Myers. This will be discussed at the end of Section **2**.

2. G-Topologies. Notation: We shall often use a description of the type of sets forming the collections G as a prefix, in place of the letter G. Thus if G is the collection of all arcs of S, we shall speak of the arc-topology of S, or the a-topology. Also for the cases where G is the collection of (1) all connected subsets of S; (2) all locally connected subsets; or (3) all rectifiable arcs, we shall use the abbreviations (1) c-topology; (2) lc-topology; or (3) r-topology.[5] The original topology of S will be referred to as the o-topology. To avoid confusion, names of topological properties will be similarly prefixed to indicate in which topology a set has the property: thus, a set may be o-connected, lc-compact, c-open, etc.

Examples: (1) If S is the closure of the graph of $y = \sin 1/x$, then the a-, c, lc-, and r-topologies are equivalent, and in any of these S is the sum of three components, two being open curves, and the third an arc.

(2) If S is the Cantor star—the join of a Cantor set and a point—then in any of the four topologies used in (1) S is, roughly speaking, the sum of uncountably many intervals all mutually perpendicular.

(3) If S is one of the locally connected spaces due to Moore [13] or to Kuratowski and Knaster [10] that contains no arc, then the c- and lc-topologies are equivalent, while the a-topology is discrete. In Moore's example,

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[4] If H is a collection of sets, H^* denotes the sum of the elements of H. A set K has the fixed-point property provided that for every continuous transformation of K into itself, some point is its own image.

[5] This notation was introduced by Wallace [19].

which is compactly connected, the compact-continuum topology is equivalent to the *lc*-topology, while in the other one, which contains no perfect set, the compact-continuum topology is discrete.

THEOREM 1. *If S is a T_0 space in its o-topology, then it is a T_0 space in any G-topology, and every o-open set is G-open. If every element of G is o-connected, then every o-component of an o-open set is G-open.*

Proof. Clearly, S and the empty set are both G-open. Let D be the sum of a collection of G-open sets, and let P be a point of one of them, say E. The statement that E is G-open implies that there is an o-open set U containing P such that no element of G lies in U and intersects both P and $S - E$. Then certainly no element of G lies in U and intersects both P and $S - D$, so that D is G-open.

The product of two G-open sets is G-open, for let D and E be two G-open sets and let P be a point of their intersection. There exist o-open sets, U and V, containing P, such that no element of G lying in U (or V) intersects P and $S - D(S - E)$. The set $U \cdot V$ is o-open and contains no element of G that intersects $S - D \cdot E$. Hence P is not a G-limit point of $S - D \cdot E$, and it follows that $D \cdot E$ is G-open. The rest of the theorem is easily proved.

THEOREM 2. *If S is a Hausdorff space in the o-topology, it is also a Hausdorff space in the G-topology.*

THEOREM 3. *If every element of G is G-connected, and S is a T_0 space, then S is G-locally connected, and if, in addition, every two points of S lie in an element of G, then S is G-connected. Hence S is a- and lc-locally connected.*

Proof. Let H be a G-component of a G-open set, D. If H is not open, it contains a point P which is a G-limit point of $D - H$. Since each element of G is G-connected, no element of G that contains P and intersects $D - H$ lies entirely in D, so that each such element intersects the G-boundary of D. But then every o-neighborhood of P contains an element of G that intersects P and the G-boundary of D, so that P is in the G-boundary of D, which is impossible. It follows that S is G-locally connected. The rest of the theorem is obvious.

It may be thought that for any collection G, of connected sets, S would be G-locally connected, but this is not the case. An example which shows this will be given following Theorem 6.

A question of considerable importance is this: Suppose that S has a strong

topology, say compact metric. By Theorem 2, for any collection G, S is Hausdorff in the G-topology. Even if S were G-locally connected, however, in general one would prefer to keep the power of compact metricity and not have the local connectivity with the weak separation axioms. Will S always have a topology that is not much weaker than its original one? For example, will S always be metric if it is originally compact metric? Unless extra conditions are imposed on G, such as those of Theorem 4, the answer to this is no. As a simple example, let S be the plane, and let G be the collection of all arcs having no more than one point in common with the x-axis. Then S is certainly not metric, nor even a Moore space in the G-topology, for if P is a point of the x-axis, the G-closure of every G-open set that contains P intersects the complement of P in the x-axis, which is G-closed.

THEOREM 4. *Let G have the property that if A is a point of an o-domain, D, there exist two o-open sets, R and R', such that* (1) *R contains R' and R' contains A, and* (2) *if C is a simple chain[6] of elements of G, one link of which contains A and all of whose links lie in R', and X is any point of C^*, then there is an element of G which lies in R and contains $X + A$. Then if S is metric or a Moore space[7] in the o-topology, it is still metric or still a Moore space in the G-topology.*

Proof. Suppose first that S is a Moore space; let $G_1, G_2, G_3, \cdots$ be the collection of o-regions postulated by Axiom 1 of Moore's book [14]. For each natural number n, let H_n be the collection of all G-open sets R defined as follows: If P is a point of an o-region U of G_n, let R be either the maximal subset of U containing P such that every two points of R can be joined by a simple chain of elements of G lying in R; or if this set is empty, let R be P. It is easy to see that R is always G-open. Then the collection $H_1, H_2, H_3, \cdots$ satisfies the first two parts of Axiom 1. Let E be a G-open set, and let A and B be points of E. To show that S is a Moore space in the G-topology, we need only show that there is a natural number n such that if h is an element of H_n that contains A, the G-closure of h is a subset of $E - B$. There is a natural number n_1 such that if R is an o-region of G_{n_1} that contains A then (1) the o-closure of R does not contain B, and (2) no element of G that

[6] A collection $H_1, H_2, \ldots, H_n$ of sets is a simple chain from the point A to the point B provided that any two successive sets have a point in common; sets that are not successive do not intersect; H_1 is the only set that contains A; and H_n is the only set that contains B. The sets H_i are the links of the chain. Cf. Moore [14], p. 56.

[7] A Moore space is a space satisfying the first three parts of Axiom 1 of Moore's book [14].

contains A and lies in R intersects $S - E$. Let D be some definite o-region of G_{n_1} that contains A, and choose two o-domains R and R' containing A and satisfying for D and A the conditions of the hypothesis. There is an integer n_2 such that the o-closure of every G-region of H_{n_2} that contains A lies in R'. Suppose that for some integer n greater than n_2 there is a G-region, F, of H_n that contains A, and whose G-closure intersects $(S - E) + B$. Then there is an element g of G that contains a point, A', of F, intersects $(S - E) + B$, and lies in R'. By the definition of $H_1, H_2, H_3, \cdots$, there is a finite chain, C, of elements of G that lies in F, and hence in K', and contains both A and A'. By our hypothesis there is, then, an element g' of G that lies in R and contains A and a point of $[(S - E) + B] \cdot g$. But this contradicts the definition of n_1. It follows that if the topology defined by the collections $H_1, H_2, H_3, \cdots$ is the G-topology, then S is a Moore space in the G-topology. But clearly our hypothesis and the definition of these collections show that this is so.

Suppose now that S is metric. We shall define a G-metric. Suppose that no distance in S is greater than 1.[8] For each two points A and B of S define the G-distance $\rho(A, B)$ as follows: (1) If there is a simple chain of elements of G from A to B, let $\rho(A, B)$ be either 1 or the greatest lower bound of the sum of the diameters of the links of such chains, whichever is the lesser; (2) if there is no such chain, let $\rho(A, B) = 1$. Let $\rho(A, A) = 0$. We need only show that this metric satisfies the triangle inequality and that it is equivalent to the G-topology. Let A, B, and C be three points. If $\rho(A, B) + \rho(B, C)$ is less than $\rho(A, C)$, there exist simple chains, H and K, of elements of G from A to B and from B to C, respectively, such that the sum of the o-diameters of the links of H and K is less than $\rho(A, C)$. But $H + K$ contains a simple chain, L, of elements of G from A to C, and the sum of the o-diameters of the links of L is less than $\rho(A, C)$, which is impossible.

Clearly, convergence in the G-topology implies convergence in the ρ metric. Suppose that β is a sequence of points converging to a point, X, in the ρ metric. Let ϵ be any positive number. There exist two o-open sets, R and R' which contain X and are of o-diameter less than ϵ, and which satisfy with respect to S and to X the conditions of the hypothesis. Let Y be any point of β such that $\rho(X, Y)$ is less than half the o-distance, d, from X to $S - R$. Then there is a simple chain, H, of elements of G from X to Y such that the o-diameter of H^* is less than d. It follows by hypothesis that there is an element of G lying in R and containing X and Y. Hence β

[8] This involves no loss of generality; any metric can be made to satisfy this by changing distances greater than 1 to 1.

G-converges to X, which completes the proof that the ρ metric and the G-topology are equivalent.

THEOREM 4(a). *Any one of the following conditions implies the condition of the hypothesis of Theorem 4:* (1) *the sum of two intersecting elements of G is an element of G;* (2) *each two points of the sum of two intersecting elements of G are joined by an element of G lying in that sum; and* (3) *if D is an o-domain, each two points of the sum of two intersecting elements of G that lie in D can be joined by an element of G lying in D. In particular, the collections of all arcs, all connected subsets and of all connected and locally connected subsets satisfy that hypothesis.*

The G-distance function, $\rho(A, B)$, cannot in general be replaced by the function $\rho'(A, B)$ defined as the greatest lower bound of diameters of elements of G containing A and B, or as 1, if none exists. We do have the following result.

THEOREM 4(b). *If S is semi-metric, for any collection G of subsets of S, the $\rho'(A, B)$ function defines a semi-metric agreeing with the G-topology of S.*

In general, even for a collection satisfying any of the conditions of Theorem 4(a) it is not true that completeness in the o-metric will imply completeness in any G-metric. A simple example of this is to let S be the plane and to let G be the collection consisting of all points of the x-axis with irrational abscissae, and of all connected sets containing no such point. The following theorem covers the most important cases.[9]

THEOREM 5. *Suppose that S is metric in its o- and G-topologies, and is o-complete. Suppose further that G is such that if $g_1, g_2, g_3, \cdots$ is a sequence of elements of G o-converging to a point P, and such that $g_1 + g_2 + \cdots$ is connected, then for every o-open set R containing P there is an integer N such that, for every $n > N$, R contains an element of G intersecting both P and g_n. Then in the G-metric, $\rho(A, B)$, S is complete.*

Proof. Suppose that $P_1, P_2, P_3, \cdots$ is a subsequence of a Cauchy sequence in the G-metric. Then this subsequence o-converges to a point P. For each integer j there is an integer N_j such that if n and m are larger than

[9] Considerable difficulty arises in the problem of G-completeness for an o-complete Moore space, due to the possible presence in $G_1, G_2, G_3, \ldots$ of "large" regions which really have nothing to do with the topology of the space. All I can say is that the collections of G-regions defined in Theorem 4 will not always satisfy part (4) of Axiom 1, though this, of course, does not imply that no set of collections will.

N_j, P_n and P_m are each at o-distance less than $1/2^j$ from P and are at G-distance less than $1/2^j$ from one another. It follows that for each two such points there is a simple chain of elements of G from P_n to P_m such that the sum of the o-diameters of its links is less than $1/2^j$; hence the sum of the chain is at o-distance less than $2/2^j$ from P. It is easy to select a subsequence $Q_1, Q_2, Q_3, \cdots$ of $P_1, P_2, P_3, \cdots$ such that for each n Q_n is at o-distance less than $1/2^n$ from P and such that there is a finite chain G_n of elements of G joining Q_n and Q_{n+1}, the sum of their o-diameters being less than $1/2^n$. The sequence of all elements of G which are links of any one of $G_1, G_2, G_3, \cdots$ clearly o-converges to P. It follows from the hypothesis and the definition of the G-topology that $Q_1, Q_2, Q_3, \cdots$ G-converges to P. Since, then, every subsequence of a G-Cauchy sequence, β, contains a G-convergent subsequence, it follows that β G-converges.

THEOREM 5(a). *If S is o-metric and o-complete, it is complete metric in each of the a-, lc-, and c-topologies.*

Proof. To take one case, if a sequence of arcs o-converges to a point P and has an o-connected sum, then this sum plus P is an o-continuous curve, and the condition of Theorem 5 is easily seen to hold.

Of particular interest for application is the question of the behavior of transformations of S under changes of topology. Of course, the continuity of a transformation of S depends not only on the topology chosen for S, but also on the topology chosen for the image of S. We shall consider, however, only one of the many possible theorems.

THEOREM 6. *If a transformation $T(S) = U$ of a T_0 space S onto a space U is o-continuous, it is G-continuous for any collection G. If further, H is a collection of subsets of U such that for any element g of G each two points of $T(g)$ are joined by an element of H lying in $T(g)$, then T is G-continuous in the H-topology of U.*

Proof. The first statement is obvious. We prove the second. Suppose that K is a subset of S and that X is a point of the G-closure of K. Since T is o-continuous, $T(X)$ is contained in the o-closure of $T(K)$ (that is, the closure of $T(K)$ in the original topology of U). If it is not a point of $T(K)$, every o-open subset R of U that contains $T(X)$ also contains the image of some element g of G that intersects X and K. By hypothesis, R then contains an element of H that intersects both $T(X)$ and $T(K)$. It follows that $T(X)$ is in the H-closure of $T(K)$, which completes the proof.

Except to give the following example, of interest in connection with the

work of Whyburn discussed below, I shall not discuss the equivalence and comparison of various types of G-topologies. This example shows that the c- and lc-topologies are not equivalent.

Example: Let T denote the plane rectangle with the points $(0, \pm 1)$ and $(1, \pm 1)$ as vertices; let R denote the closure of the set of points of the graph of $y = \sin \pi/x$ for which $0 < x < 1$; let P denote the point $(1/2, 0)$ and Q denote the point $(1/3, 0)$. For each positive integer n, let R_n, P_n, and Q_n denote the images of R, P, and Q, respectively, under a sense-preserving affine transformation of the plane throwing T into the rectangle whose vertices are $(1/(n+1), \pm 1/n)$, $(1/n, \pm 1/n)$; and let A_n denote the arc joining Q_n to P_{n+1} consisting of two vertical intervals and an interval of the line $y = 1$. Let I denote the interval $(0, 0)$, $(0, 1)$ of the y-axis, and let S denote $I + A_1 + R_1 + A_2 + R_2 + \cdots$, topologized by the plane metric. Then in the lc- or a-topologies the sets I and $S - I$ are mutually separated, while in the c-topology S is still connected.

The viewpoint of this section has some precedence in the literature. Mazurkiewicz [11] defined and studied for compact locally connected spaces, K, a metric which replaced the distance between each two points of K by the greatest lower bound of diameters of continua containing the two points. Whyburn [24] (see also page 154 et seq. of his book [25]) extended this definition to general locally connected metric spaces by replacing the requirement that the sets defining the new metric be continua with the requirement that they be merely connected, pointing out that Mazurkiewicz' definition was inadequate if the spaces studied were not compactly connected. Whyburn exploited this metric—his relative distance transformation—primarily in studying plane domains.[10] From the standpoint of this paper, Mazurkiewicz was studying the compact-continuum topology and Whyburn the c-topology of their respective spaces, and using essentially the metric of Theorem 4. However, in each case, though the metric changed, the topology of the space did not. From another approach, ideas similar to those of this paper have been developed in metric geometry by Menger [12] and Myers [15]. For compact metric spaces M in which each two points are joined by an arc of finite length (and hence by a geodesic [11]) Menger defined a metric as the length of the geodesic joining each pair of points. In this geodesic metrization, M is convex, but not necessarily topologically the same. Myers

[10] Some of the general properties of his metric are valid in the more general spaces of this paper, but I have not explicitly stated these.

[11] For a discussion of these and related topics, see Chapter VI of Blumenthal's book [3].

extended this to rectifiably connected metric spaces in general by using for the new metric the greatest lower bound of lengths of arcs joining two points. He considered many of the properties of such spaces, proving results like many of those considered here, and using the properties of the new metric to obtain conditions for the existence of geodesics. His results are not special cases of those herein; if G is the collection of all rectifiable arcs, the G-metric is expressed in terms of the diameters and not of the lengths of these arcs, so that the G-metric can become small without the geodesic metric decreasing. A unified treatment is of course possible, but it seemed to me that the rather slight gain was not worth the extra space required. I should also mention the work of Hewitt [5] who in other connections used the same general idea of change of topology as a method of proof.

3. Generalized dendrites. In this section, I shall define a type of generalized dendrite and study its properties. I am particularly interested in deriving a certain condition for the fixed-point property to hold, but it will become clear that the methods used will permit a number of theorems involving mappings of dendrites to be proved for at least many generalized dendrites. It would be possible to shorten the length of this section by assuming that we are dealing with a metric space. Though this would not greatly affect the use I intend to make of the results of this section to illustrate the use of the preceding section, it seems preferable to obtain as much generality as possible.

Definition: By a generalized dendrite [12] is meant a locally connected Hausdorff space T such that if A and B are two points of T, and Δ_1 and Δ_2 are two simple chains of connected domains from A to B, and Δ_1 has more than two links, then some link of Δ_1 that does not contain A or B intersects some link of Δ_2. Throughout this section T denotes such a space.

THEOREM 7. *Under the hypotheses of the definition every link of* Δ_1 *intersects some link of* Δ_2.

Proof. Suppose that $\Delta_1 = D_1, D_2, D_3, \cdots$, where the numbering is in the order from A to B, and suppose that for some integer j, D_j intersects no link of Δ_2. Then there exist integers i and k such that $i < j < k$, and D_i and D_k intersect links of Δ_2, but no link of Δ_1 between D_i and D_k intersects a link of Δ_2. There exist two points, X and Y, in D_i and D_k, respectively such

[12] This is not the same as the generalized dendrite recently defined by Shanks [17], which is a generalization removing local connectivity. My generalization is along the line of Moore's dendron [14], Whyburn's denodular sets [22], and Wallace's trees [18]. In each of the spaces considered by these authors, their term is equivalent to mine.

that Δ_1 and Δ_2 contain subchains from X to Y. But clearly the existence of these subchains contradicts the definition of T.

THEOREM 8. *If the closed set H separates the point A from the point B in T then some point of H separates A from B in T.*

Proof. Let C_1 be the component of $T - H$ containing A and let C_2 be the component of $T - (\bar{C}_1 - C_1)$ that contains B. Suppose that the boundary of C_1 contains two points, X and Y. There exist mutually exclusive connected domains, D and E, containing X and Y, respectively. Then C_1, D, and C_2, and C_1, E, and C_2 are two chains contradicting the definition of T.

THEOREM 9. *If A and B are two points of T, they are the non-cut points of a bicompact and (sequentially) compact, locally connected, connected set, H, such that each point of $H - (A + B)$ separates A from B in T.*

Proof. Let H denote the set of all points P such that if Δ is a simple chain of connected domains from A to B, then Δ^* contains P. If the point X of $\bar{H} - (A + B)$ does not separate A from B, there is a simple chain, Δ, from A to B of connected domains no one of which contains X in its closure. Then $\overline{\Delta^*}$ contains H, but not X. Hence every point of $\bar{H} - (A + B)$ separates A from B, and therefore belongs to H. We now show that H is bicompact. Let G be any collection of domains covering H. Let G' denote the collection of components of $T - H$; there is a simple chain of connected domains from A to B every link of which is either a domain of G' or a component of a domain of G. Bicompactness follows readily.

If H is not connected, then there exist two points, X and Y, of H such that no point of H is between them in the separation ordering of H. There is a domain D containing X such that $\bar{D} \cdot Y = 0$. Then $\bar{D} - D$ separates X from Y in T. By Theorem 8, so does some point, Z, of $\bar{D} - D$. But then Z belongs to H, which is impossible.

The other properties of H follow from the results of F. B. Jones [7], since it is easy to see that H is linear in his sense, or they are not hard to prove directly.

THEOREM 9(a). *If T is separable, or is a Moore space, the set H of Theorem 9 is a true arc.*

Proof. The argument used in the proof of Theorem 6 of my paper [26] can be modified to prove that if L is a closed subset of a locally connected Hausdorff space, M, and K is a countable set of points of M dense in L such that each point of K either belongs to L or is separated from some point of

L by some other point of L, then L is separable. Hence H is separable, and, by Theorem 10 of Jones [7], is a true arc. For a Moore space, this is a consequence of Theorem 2, p. 432, of Moore [14].

THEOREM 10.[13] *Every connected subset of T is itself a generalized dendrite.*

Proof. If C is a connected, locally connected subset of T, clearly it follows from Theorem 9 that it is a generalized dendrite. Suppose that the connected subset C is not locally connected at a point A. There is a connected domain, R, of T containing A, such that $C \cdot R$ is not connected. From the argument in Theorem 9 each two points of R are joined by a "pseudo arc" lying in R. If its ends lie in C, so does the "pseudo arc." This gives a contradiction.

THEOREM 11. *If T is either* (1) *bicompact and separable; or* (2) *a separable Moore space having only a countable number of non-cut points, then T is metric.*

Proof. We shall show that T is metric under the first hypothesis. If K is a countable subset of T dense in T, select for each two points of K a point of T that separates them; let K' denote the set of all such points. Clearly K' is countable and dense in T. Let G be the collection of all connected open sets whose boundaries are finite subsets of K'; since there are only a countable number of finite subsets of K', it readily follows that G is countable. Let P be a point and R be a connected open subset of T containing P. For each point X of the boundary of R choose a point P_X of K' separating X from P; let D_X be the component of $T - P_X$ that contains X. Since $T - R$ is closed, a finite collection of domains D_X covers it. Let H denote the sum of their boundaries. Then H is a finite subset of K' and the component of $T - H$ that contains P lies together with its boundary in R. Consequently, T is regular in the sense of Alexandroff and Urysohn [25] and is completely separable; therefore it is metric.

In the second case, by a theorem of Jones [6], if every uncountable subset of T has a limit point, then T is metric. If K is an uncountable subset of T, the argument used in Theorem 72 of Chapter I of Moore [14] proves that some two points of T are separated by uncountably many points of K. It follows from Theorem 9(a) that K has a limit point.

Neither of the hypotheses of Theorems 9(a) or 11 can be weakened by the omission of any condition and still have the theorems remain true. On

[13] Cf. Theorem 32, p. 116, of Moore [14].

the other hand the condition of separability alone lets us strengthen the topology of T into a metric space.

THEOREM 12. *If T is separable, then there is a biunivalued continuous transformation of T into a connected subset of a compact, metric dendrite. If in addition the sum of every monotone increasing sequence of arcs of T is contained in an arc of T, then this subset is closed.*

Proof. The argument given by Whyburn in [21] to prove a similar theorem for metric, separable potentially regular sets, and which is closely related to the proof of Theorem 11, is sufficient under our hypotheses to prove that T can be mapped in the desired way onto a metric generalized dendrite. The proof consists essentially in redefining "neighborhood" to mean a complementary domain of a finite subset of the set K' defined in Theorem 11, and proceeding as in that theorem to prove that T is metric in this new topology. Then either Theorem A_1 of Whyburn's [21] or Theorem 8.2 of his [22] shows that T can be mapped onto a subset M of a compact dendrite D by a transformation $f(T) = M$ of the desired type.

To prove the rest of the theorem, let X be a point of $\bar{M} - M$, if any exists. Then there is a sequence, $X_1, X_2, X_3, \cdots$ of points of M approaching X such that for each n X_n separates X_{n-1} from X in $\bar{M}$. The sequence X_1X_2, $X_1X_3, X_1X_4, \cdots$ of arcs of M is monotone increasing, and hence so is $f^{-1}(X_1X_2), f^{-1}(X_1X_3), \cdots$. In T there is an arc AB irreducibly containing all these arcs. Then $f(A + B) = X + X_1$, which proves that M is closed.

We are now in a position to prove the fixed-point theorem for generalized dendrites mentioned above. This theorem will be stated for a generalized dendrite which is a Hausdorff space, but the proof will be valid only for arcwise-connected generalized dendrites. The reason for this is that it is possible to give a proof for the general case merely by modifying the argument for Theorem 2.1 of Chapter XII of Whyburn's book [25], closed, connected subsets taking the place of the A-sets, and the existence of certain intersections being guaranteed by the condition of the theorem. But such a proof would have little novelty, and I prefer to give the more restricted one here, which indicates a general method for studying transformation properties of certain generalized dendrites by means of Theorem 12.

THEOREM 13.[14] *A sufficient condition that a generalized dendrite T have*

[14] Cf. Wallace [18], where he proves a similar theorem for certain point-to-set mappings of bicompact Hausdorff trees. My result generalizes his for point-for-point mappings. I have not obtained a full generalization.

the fixed-point property is that the sum of any increasing simple sequence of pseudo arcs of T be contained in a pseudo arc.

Proof. Assume T to be arcwise connected. Let $f(T)$ be a continuous transformation of T into itself. Let A be any point of T. If $f(A) = B$, where B is not A, then the image of the arc AB, $f(AB)$, is a compact continuous curve intersecting AB in at least B. Indeed, for each positive integer n, $f^n(AB)$ is a compact continuous curve intersecting $f^{n-1}(AB)$. Hence the semi-orbit, H, of AB is a completely separable, connected subset of T, and $\bar{H}$ is a separable subcontinuum of T which is mapped into itself by f. There is a mapping $g(H)$ onto a compact dendrite K satisfying the conditions of Theorem 12, since clearly every increasing sequence of arcs of $\bar{H}$ is contained in an arc of $\bar{H}$. The mapping $gfg^{-1}(K)$ is a continuous mapping of K into itself, for suppose that $X_1, X_2, X_3, \cdots$ is a sequence of points of K converging to a point X. Then it is easy to see that no point of $\bar{H}$ separates $g^{-1}(X_1 + X_2 + X_3 + \cdots)$ from $g^{-1}(X)$ in $\bar{H}$; and hence that no point of $\bar{H}$ separates $fg^{-1}(X_1 + X_2 + \cdots)$ from $fg^{-1}(X)$. It follows that no point of K separates $gfg^{-1}(X_1 + X_2 + \cdots)$ from $gfg^{-1}(X)$ in K; and since this is also true for any subsequence of $X_1, X_2, \cdots$, that $gfg^{-1}(X)$ is a point or a limit point of $gfg^{-1}(X_1 + X_2 + \cdots)$. It follows from the Scherrer Theorem—Theorem 3.21 of Chapter XII of [25]—that there is a point Y of K such that $gfg^{-1}(Y) = Y$. Then $fg^{-1}(Y) = g^{-1}(Y)$, which proves the theorem.

THEOREM 13(a). *If T is arcwise connected, the condition of Theorem 13 is also necessary.*

Proof. If there is a sequence of arcs of T not satisfying our condition, then it follows that there is a ray R in T. There is a retraction $f(T) = R$ throwing each component of $T - R$ into its boundary point, and there is a "translation" of R, $g(R) = R'$, where R' is a proper subray of R. Then the transformation $gf(T)$ is continuous and leaves no point fixed.

That the condition of Theorem 13 does not imply local compactness or separability, even if T is metric, can be seen by letting T denote the set of all points in or on the unit square, with distance between two points defined as the length of the arc joining the points composed of vertical intervals and, if necessary, an interval of the x-axis. Nor, in general, does it even imply the existence of two non-cutpoints, as will be seen below. Also in general, the condition is not necessary. If α is any ordinal greater than 1, by an α-arc I shall mean the set obtained by taking the sequence of ordinals up to and including α and inserting an open line segment between each two consecutive

ordinals, with the topology defined in a natural way by the order. In this order topology, an α-arc is a bicompact Hausdorff space,[15] and is a tree in the sense of Wallace [18], and hence has the fixed-point property. The set obtained from an α-arc by deleting α, I shall call an α-ray. Some α-rays have no countable sequence running upward through them, and hence have the fixed-point property by Theorem 13, but there are still others that have the property.

THEOREM 14. *If the α-ray R does not contain an increasing sequence $\alpha_1, \alpha_2, \alpha_3, \cdots$ such that no point of R follows all the points of the sequence and such that for $n = 2, 3, \cdots$ the α-arc $\alpha_n\alpha_{n-1}$ is of the same or lower cardinal than $\alpha_{n-1}\alpha_n$, then R has the fixed-point property.*

Proof. Let P be the ordinal 1, and let $T(R)$ be a continuous transformation of R into itself. If P is not fixed, let $T(P) = P'$. There is a point Q of R which is the first point to follow all the points of $T(PP')$, $T^2(PP')$, $T^3(PP'), \cdots$, since otherwise there is a sequence contradicting the condition consisting of one point from each iterate of PP'. The pseudo arc PQ is mapped into itself by T, which, with Wallace's theorem, or Theorem 13, completes the proof.

As another example of the use of Theorem 12, I shall state without proof—which can easily be supplied—a partial extension of a theorem of Schweigert's [16], which has been generalized by Wallace [20].

THEOREM 15. *If T is a generalized dendrite satisfying the condition of Theorem 13, $f(T) = T$ is a homeomorphism, and P is a point of T which is fixed under f and which is an end point of every arc containing it,*[16] *then there is another point fixed under f.*

4. A fixed-point theorem. In this section we shall combine the results of Sections **2** and **3** to prove that certain connected sets which exhibit dendritic properties have the fixed-point property, even though they are not locally connected. The class of sets we shall consider are the arcwise connected sets such that each increasing sequence of arcs is contained in an arc. Following are several examples of spaces having this property:

a) The sum of the unit interval and a collection of intervals perpendicular to this interval at the points of irrational abscissae.

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[15] Cf. Theorem 2.14, p. 28, of [2].

[16] This is not the same as requiring that P be a point of Menger order 1. See the example following Theorem 13.

b) The Cantor star.

c) The continuum obtained by joining the points of a hereditarily indecomposable plane continuum to a point not in the plane by straight-line intervals.

That example b) has the property follows immediately from the result of Hamilton [4] that every hereditarily unicoherent, hereditarily decomposable compact continuum has the fixed-point property, or from the result of Kelley [9] concerning simple links. But neither of these results applies to the other two examples.

THEOREM 16. *Let M be an arcwise connected Hausdorff space which is such that every monotone increasing sequence of arcs is contained in an arc. Then M has the fixed-point property.*

The theorem in the introduction is an immediate consequence.

Proof. In its o-topology, and hence in its a-topology, M contains no simple closed curve; hence in the a-topology it is a generalized dendrite such that every monotone increasing sequence of arcs is contained in an arc. Further, by Theorem 12, every o-continuous mapping of M into a subset, H, of itself is a-continuous in either of the topologies of H determined by the o- or a-topology of M. Hence by Theorem 13, M has the fixed-point property.

The same argument proves from Theorem 15 the following result.

THEOREM 17. *If M is defined as above, and $f(M) = M$ is a homeomorphism, and P is a point fixed under f which is an end point of every arc of M containing it, then there is another point of M fixed under f.*

For the three examples given above, this result does not follow from Wallace's generalization of Schweigert's theorem.

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ON THE SOLUTIONS OF AN ORDINARY DIFFERENTIAL EQUATION NEAR A SINGULAR POINT.*

By PHILIP HARTMAN.

This paper is a continuation of the considerations of a recent paper by Wintner and the present author [3] dealing with Perron's [4] generalization of certain of Poincaré's qualitative results on the singularity $(x, y) = (0, 0)$ of the real (analytic) differential equation

$$xy' = \alpha x + \beta y + \cdots \qquad (\beta \neq 0) \tag{1}$$

and the corresponding results of Bendixson on the non-analytic equation

$$x^m y' = \alpha x + \beta y + \cdots \qquad (\beta \neq 0,\ m = 1, 2, \cdots)$$

(cf. Liebmann, [2], pp. 507-512 and Dulac [1], p. 178 for further references).

Perron dealt with the differential equation

$$\phi(x) y' = f(x, y), \tag{2}$$

where $\phi(x)$ is any positive continuous function on an interval $0 < x \leqq a$ satisfying

$$\phi(x) \to 0 \text{ as } x \to +0 \tag{3}$$

and

$$\int_{+0} dx/\phi(x) = +\infty, \tag{4}$$

and where $f(x, y)$ is a real-valued continuous function on a closed rectangle $0 \leqq x \leqq a$, $|y| \leqq b$ subjected to the condition that

$$f(0, 0) = 0 \tag{5}$$

and to the upper and lower Lipschitz conditions

$$|(f(x, y_1) - f(x, y_2))/(y_1 - y_2)| < C, \qquad (y_1 \neq y_2), \tag{6}$$

$$|(f(x, y_1) - f(x, y_2))/(y_1 - y_2)| > c > 0, \qquad (y_1 \neq y_2). \tag{6 bis}$$

Perron showed that there are two different situations depending on the algebraic sign of the non-vanishing difference quotient in (6 bis). If this

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sign is plus, there exist two positive numbers a', b' such that every solution path $y = y(x)$ of (2) issuing from a point of the rectangle

$$R' : 0 < x < a', \mid y \mid < b' \qquad (a' \leqq a, b' \leqq b) \tag{7}$$

satisfies

$$\lim_{x \to +0} y(x) = 0. \tag{8}$$

If the sign is minus, there exists exactly one solution path $y = y(x)$ satisfying (8).

In [3], it was shown that Perron's analytic proof involving successive approximations could be avoided by the use of simple geometrical considerations, and at the same time, the heavy conditions imposed by Perron could be lightened to a great degree. The principal theorem proved there was:

(*) *Let $\phi(x)$, where $0 < x \leqq a$, be a positive, continuous function satisfying* (4). *Let $f(x, y)$ be defined and continuous on the (partly open) rectangle*

$$R: 0 < x \leqq a, \mid y \mid \leqq b, \tag{9}$$

having the property that $1/f(x, y)$, outside of any fixed vicinity of the origin, is bounded for small x and that $f(x, b)$ and $f(x, -b)$ do not vanish for $0 < x \leqq a$ and are of opposite signs.

(I) *if, further, $f(x, b) > 0$ and $f(x, -b) < 0$ for $0 < x \leqq a$, then every solution path $y = y(x)$ of* (2) *issuing from a point (x_0, y_0) of the rectangle* (9) *can be continued for all positive $x (< x_0)$ and* (8) *holds for all continuations.*

(II) *If, however, $f(x, b) < 0$ and $f(x, -b) > 0$ for $0 < x \leqq a$, then there exists at least one solution path $y = y(x)$ of* (2) *defined for all small positive x and* (8) *holds for all such paths.*

(It may be remarked, for completeness' sake, that the solution in (II) is unique, if $f(x, y)$ is, for every fixed x, monotone with respect to y).

In addition to the limit relation (8), Perron also considered the behavior of the ratio $y(x)/x$. Under the conditions imposed by him on $f(x, y)$, there exists, for small $x \geqq 0$, a unique continuous function $y = y^0(x)$ such that $y^0(0) = 0$ and

$$f(x, y^0(x)) \equiv 0. \tag{10}$$

Assuming further that

$$\lim_{x \to +0} y^0(x)/x = \gamma \text{ exists}, \tag{11}$$

and that

$$\phi(x)/x \to 0, \text{ as } x \to +0, \tag{12}$$

Perron showed that for any solution path $y = y(x)$ of (2) satisfying (8),

$$\lim_{x \to +0} y(x)/x = \gamma. \tag{13}$$

The object of this paper is to obtain similar relations under improved conditions by the use of the geometrical arguments employed in [3]. (For another investigation of similar asymptotic statements in the non-analytic case of (1), cf. Wintner [6]; the results of this paper are in a different direction, however.)

In the case of the above italicized theorem (*), it turned out that Perron's (upper) Lipschitz condition (6) was entirely superfluous, while the (lower) condition (6 bis) could be considerably weakened to conditions on $f(x, y)$ near the boundary lines $x = 0$, $y = +b$, $y = -b$. In the present case, the situation is similar in that (6) is not needed, while (6 bis) is retained either in its full strength or, in some instances, in a weakened form.

First, the following theorem will be proved:

THEOREM 1. *Let $\phi(x)$ be a positive, continuous function on the interval $0 < x \leqq a$ satisfying* (12). *Let $f(x, y)$ be a real-valued, continuous function on the partly open rectangle R,* (9), *and satisfy there the lower Lipschitz condition* (6 bis). *Let there exist a function $y = y^0(x)$, $0 < x \leqq a$, satisfying $|y^0(x)| < b$, and* (10) *and* (11). *If $y = y(x)$ is any solution path of* (2) *satisfying* (8), *then* (13) *holds.*

Proof. It may be remarked that as far as the existence of solutions $y = y(x)$ of (2) satisfying (8) is concerned, the conditions of the theorem (*) above are satisfied. In fact, for the applicability of (*), $y^0(x) \to 0$ as $x \to +0$ could replace (11).

It may be supposed that the constant c in (6 bis) is 1; otherwise the differential equation (2) can be divided by c and this factor absorbed in both $f(x, y)$ and $\phi(x)$ without changing either the conditions or the statement of the theorem. Also, in order to fix ideas, it will be supposed that the algebraic sign of the non-vanishing difference quotient in (6 bis) is plus. The proof in the other case is similar. Thus, (6 bis) becomes

$$f(x, y_1) - f(x, y_2) > y_1 - y_2 \quad \text{if } y_1 > y_2, \tag{14}$$

for $0 < x \leqq a$.

Introducing the new dependent variable

$$v = v(x) = y(x)/x, \tag{15}$$

the differential equation (2) becomes

$$\phi(x)v' = F(x, v), \tag{16}$$

where

$$F(x, v) = [f(x, vx) - v\phi(x)]/x \tag{17}$$

is continuous on its domain of definition $0 < x \leqq a$, $| v | \leqq b/x$. This function satisfies the inequality

$$F(x, v_1) - F(x, v_2) > (v_1 - v_2)(1 - \phi(x)/x) \text{ if } v_1 > v_2. \tag{18}$$

It follows from (12) that for sufficiently small positive x, $F(x, v)$ is monotone increasing in v.

It will be shown *that if x is sufficiently small, there exists a unique value of $v = v^0(x)$ such that $F(x, v^0(x)) = 0$ and that $v^0(x)$ lies between $y^0(x)/x$ and $y^0(x)/(x - \phi(x))$.* That $v^0(x)$ is unique follows from the strict monotony of $F(x, v)$ with respect to v. Now, place $v = y^0(x)/x$ in the definition (17) of $F(x, v)$; by (10)

$$F(x, y^0(x)/x) = -y^0(x)\phi(x)/x^2$$

which either vanishes (together with $y^0(x)$) or has the same sign as $-y^0(x)$. If $y^0(x) = 0$, nothing remains to be proved. Suppose, for a moment, that $y^0(x) > 0$ for this fixed x; so that $xy^0(x)/(x - \phi(x)) > y^0(x)$ for small $x > 0$. Subtracting $0 = f(x, y^0(x))/x$ from the function (17), with v replaced by $y^0(x)/(x - \phi(x))$, and applying (14) to $y_1 = xy^0(x)/(x - \phi(x))$ and $y_2 = y^0(x)$, one obtains

$$F(x, y^0(x)/(x - \phi(x)) > 0.$$

Similarly, if it is supposed that $y^0(x) < 0$, it follows that $F(x, y^0(x)/(x - \phi(x))) < 0$. In view of the continuity of $F(x, v)$, this concludes the proof of the italicized statement concerning $v^0(x)$.

It follows from (11) and (12) that

$$v^0(x) \to \gamma \text{ as } x \to +0. \tag{19}$$

From (18) and (19) it also follows that if a' is any sufficiently small positive number and if $b' = b/a'$, then $F(x, b') > 0$ and $F(x, -b') < 0$ for $0 < x \leqq a'$ and that $1/F(x, v)$, outside of any neighborhood of the point $(x, v) = (0, \gamma)$, is bounded for small $x > 0$ (when $| v | \leqq b'$). The proof of (*) implies that if $v = v(x)$ is any solution path of (16) through a point (x_0, v_0) of the rectangle $0 < x_0 \leqq a'$, $| v_0 | \leqq b'$, then $v(x)$ can be continued for all positive $x (< x_0)$ and for all continuations

$$v(x) \to \gamma \text{ as } x \to +0. \tag{20}$$

In view of the fact that every solution path $y = y(x)$ of (2) satisfying (8) corresponds to a solution $v = v(x)$ of (16) and that $|v(x_0)| = |y(x_0)/x_0| \leqq b/x_0$ for all $x = x_0$, it can be concluded, by choosing $a' = x_0$ (and $b' = b/a'$) for sufficiently small x_0, that (20) holds.

By (15), this proves Theorem 1.

From the proof of this theorem and from the proof of (*), the following corollary may be deduced:

COROLLARY 1. *Let $\phi(x)$ be a positive, continuous function in the interval $0 < x \leqq a$ such that*

$$\limsup_{x \to +0} \phi(x)/x = \lambda < +\infty . \tag{21}$$

Let $f(x, y)$ satisfy the conditions of Theorem 1. *Let*

$$\lambda < c, \tag{22}$$

where $c > 0$ is a constant satisfying (6 bis). *If $y = y(x)$ is any solution path of* (2) *satisfying* (8), *then* $\liminf y(x)/x$ *and* $\limsup y(x)/x$ *lie between γ and $c\gamma/(c - \lambda)$.*

The proof of this corollary merely depends on the fact that, in the proof of Theorem 1, where it was supposed that $c = 1$, one can conclude only that $\limsup v^0(x)$ and $\liminf v^0(x)$ lie between γ and $\bar{\gamma}/(1 - \lambda)$ if (21) replaces (12).

If the limit γ in (11) is zero, this corollary becomes:

COROLLARY 2. *Let $\phi(x)$ be a positive, continuous function on the interval $0 < x \leqq a$ satisfying* (21). *Let $f(x, y)$ satisfy the conditions of Theorem* 1 *with* (11) *particularized to*

$$\lim_{x \to +0} y^0(x)/x = 0. \tag{11 bis}$$

Let (22) *be satisfied. If $y = y(x)$ is any solution path of* (2) *satisfying* (8), *then*

$$\lim_{x \to +0} y(x)/x = 0. \tag{13 bis}$$

It may be remarked that condition (6 bis) in Theorem 1 cannot be replaced by the condition that $f(x, y)$ be strictly monotone with respect to y for a fixed $x > 0$ (a condition which suffices for (*)). This may be seen from the example

$$x^2 y' = y^3$$

where the solutions are $y \equiv 0$ or $y = [x/2(1 + Kx)]^{\frac{1}{2}}$, K being an arbitrary

integration constant. Thus (13 bis) fails to hold except for the solution $y \equiv 0$ even though (11 bis) and (12) are true.

Nevertheless, Corollary 2 can be improved in some directions; the first extension [1] corresponding to the case (II) of (*) is:

THEOREM 2. *Let $\phi(x)$ and $f(x, y)$ satisfy the conditions of (*) in the case* (II). *Let $y^+(x)$, $y^-(x)$ denote, for a fixed $x > 0$, the greatest and least value of y, $|y| < b$, for which $f(x, y) = 0$, and let*

$$\lim_{x \to 0} y^+(x)/x = \lim_{x \to 0} y^-(x)/x = 0. \tag{23}$$

Then (13 bis) *holds for any solution path $y = y(x)$ of* (2) *satisfying* (8).

Proof. Let $\epsilon > 0$ be fixed and let $\delta = \delta(\epsilon)$ be so small that

$$|y^+(x)| < \epsilon x \text{ and } |y^-(x)| < \epsilon x \text{ if } 0 < x < \delta.$$

Then, if $y = y(x)$ is any solution path of (2) satisfying (8),

$$|y(x)| < \epsilon x \text{ if } 0 < x < \delta. \tag{24}$$

For suppose, if possible, that for some $x = \bar{x}$, $0 < \bar{x} < \delta$,

$$y(\bar{x}) \geqq \epsilon \bar{x} > 0.$$

In particular $y(\bar{x}) > y^+(\bar{x})$, and so from the definition of y^+, the continuity of $f(x, y)$ and the fact that $f(x, b) < 0$, it follows that $f(\bar{x}, y(\bar{x})) < 0$. From (2), it is seen that $y'(\bar{x}) < 0$ and so $y(x)$ increases with decreasing x at $x = \bar{x}$. This situation must pertain for all x, $0 < x \leqq \bar{x}$. Since $y(\bar{x}) > 0$, the relation (8) cannot hold. This contradiction proves (24).

This completes the proof of Theorem 2.

For the corresponding case (I) of (*), the above example indicates that more stringent conditions are needed. There will now be proved: [2]

THEOREM 3. *Let $\phi(x)$ and $f(x, y)$ satisfy the conditions of (*) in the case* (I). *Let, in addition,*

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[1] In [5], Perron proved a similar theorem, namely: *If $f(x, y)$ $g(x, y)$ are continuous on the closed rectangle $0 \leqq x \leqq a$, $|y| \leqq b$ and $f(x, y) = o(|x| + |y|)$, $g(x, y)$ $o(|x| + |y|)$ as $x, y \to 0$, then there exists at least one solution path $y = y(x)$ of*

$$[x + g(x, y)]dy = -c[y + f(x, y)]dx, \qquad c > 0,$$

such that $y(x) \to 0$ and $y'(x) \to 0$ as $x \to +0$.

[2] This theorem is an improvement over that of Perron [5], Satz 3, p. 129: *If $g(x, y)$ is continuous on the closed rectangle $0 \leqq x \leqq a$, $|y| \leqq b$ and $g(x, y) = o(x + |y|)$ as $x, y \to 0$, then for any solution $y = y(x)$ of*

$$xy' = cy + g(x, y), \qquad c > 1,$$

satisfying (8), *one has* (13 bis).

$$|f(x,y)| > c\,|y| + o(x) \text{ as } x, y \to 0 \qquad (c > 0). \tag{25}$$

Finally, let $\phi(x)$ *satisfy* (21) *and let* (22) *be satisfied. Then, for any solution path* $y = y(x)$ *of* (2), (13 bis) *holds.*

Proof of Theorem 3. If $y^+(x)$ and $y^-(x)$ have the same meaning as in Theorem 2, (25) implies (23).

Now, introduce the dependent variable (15), so that (2) becomes the equation (16). Let y be a fixed number such that $y > \max(0, y^+(x))$. The function $xF(x, y/x) = f(x,y) - y\phi(x)/x$ is not less than

$$y(c - \phi(x)/x) + o(x) \qquad \text{as } x, y \to 0 \tag{26}$$

by (25). From (21) and (22), it is seen that (26) is a positive quantity for sufficiently small $x > 0$. Similarly, if $y < \min(0, y^-(x))$, then $xF(x, y/x)$ is not greater than (26), which is negative for sufficiently small $x > 0$. Consequently, there exist two positive numbers a', b' $(a' \leqq a,\ b' \leqq b)$ such that for every x, $0 < x \leqq a'$, there exists at least one value of y, $|y| \leqq b'$, for which $xF(x, y/x) = 0$; and if $y_1^+(x)$, $y_1^-(x)$ denotes the greatest and least such values of y, then

$$y_1^+(x)/x \to 0, \qquad y_1^-(x)/x \to 0, \text{ as } x \to +0.$$

Hence, for every x, $0 < x \leqq a'$, there exists at least one value of v, $|v| \leqq b'/x$, for which $F(x, v) = 0$; and if $v^+(x)$, $v^-(x)$ denote the greatest and least such values of v, then

$$v^+(x) \to 0,\ v^-(x) \to 0 \text{ as } x \to +0$$

(actually $v^+ = y_1^+/x$; $v^- = y_1^-/x$).

Also, from (17) and (25),

$$|F(x,v)| > (c - \phi(x)/x)\,|v| + o(x) \text{ as } x, |vx| \to 0.$$

The proof of Theorem 3 can now be completed with the same arguments used in the proof of Theorem 1.

These two theorems yield the following corollaries:

COROLLARY 3. *Let* $f(x,y)$ *be a continuous function on the partly open rectangle* R, (9), *and let*

$$f(x,y) = \alpha x + \beta y + o(x + |y|) \text{ as } x, y \to 0 \tag{27}$$

where

$$\beta < 0. \tag{28}$$

Let $\phi(x)$ be a positive, continuous function on the interval $0 < x \leqq a$ and let

$$\phi(x)/x \to \lambda \text{ as } x \to +0. \tag{29}$$

If $y = y(x)$ is any solution of (2) satisfying (8), then

$$\lim_{x \to +0} y(x)/x = -\alpha/(\beta - \lambda). \tag{30}$$

COROLLARY 4. *Let $f(x, y)$ be a continuous function on the partly open rectangle R, (9), and let (27) hold, but*

$$\beta > 0.$$

Let $\phi(x)$ satisfy the conditions of Corollary 3 and

$$\lambda < \beta. \tag{31}$$

Then there exist two positive numbers a', b' such that if $y = y(x)$ is a solution of (2) issuing from a point (x_0, y_0) of the rectangle R', (7), then $y = y(x)$ can be continued for all positive $x (< x_0)$ and (30) holds for any such continuation.

Proof of Corollaries 3 and 4. Introducing the new dependent variable

$$w = w(x) = y(x) + \alpha x/(\beta - \lambda),$$

the differential equation (2) becomes

$$\phi(x) w' = g(x, w), \tag{32}$$

where the function

$$g(x, w) = f(x, w - \alpha x/(\beta - \lambda)) + \alpha\phi(x)/(\beta - \lambda)$$

is continuous on the partly open parallelogram $0 < x \leqq a$, $|w - \alpha x/(\beta - \lambda)| \leqq b$. From (27) and (29),

$$g(x, w) = \beta w + o(x + |w - \alpha x/(\beta - \lambda)|)$$

as $x \to 0$ and $w - \alpha x/(\beta - \lambda) \to 0$, which implies

$$g(x, w) = \beta w + o(x + |w|) \text{ as } x, w \to 0.$$

Corollaries 3 and 4 now follow by an application of Theorems 2 and 3, respectively, to the equation (32).

If conditions of the type (21) and (22) or (29) and (31) on $\phi(x)$ are weakened to allow the "$<$" to be replaced by "$\leqq$," it is easy to see that in general no statement concerning the existence of the limit of $y(x)/x$ can be made. However, even without assumptions on the zeros of $f(x, y)$, one can

obtain results about the ratio $(y_1(x) - y_2(x))/x$, where $y = y_1(x)$ and $y = y_2(x)$ are two solutions of (2). The first of such statements is:

THEOREM 4. *Let $\phi(x)$ and $f(x, y)$ satisfy the conditions of* (*) *in the case* (I). *Let, in addition, $f(x, y)$ satisfy the lower Lipschitz condition* (6 bis). *Let $\phi(x)$ satisfy*

$$c/\phi(x) - 1/x \geqq 0 \text{ for small } x > 0, \tag{33}$$

where c is a constant satisfying (6 bis). *Then, if $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of* (2) *satisfying* (8),

$$\lim_{x \to +0} (y_1(x) - y_2(x))/x \text{ exists} \tag{34}$$

($\pm \infty$ are allowed in (34)).

Proof. Again assume, for simplicity, that $c = 1$, so that (6 bis) becomes (14). Introduce the dependent variable (15), so that the equation (2) becomes (16).

Let $y = y_1(x)$, $y = y_2(x)$ be two solutions of (2) satisfying (8). It may be supposed that $y_1(x) \geqq y_2(x)$ for all small x (otherwise the indices may be interchanged for some x). If $v_1 = y_1(x)/x$, $v_2 = y_2(x)/x$, it follows from (16) and (18) that

$$(v_1 - v_2)' \geqq (v_1 - v_2)[1/\phi(x) - 1/x]. \tag{36}$$

From (33) it is seen that $(v_1 - v_2)' \geqq 0$, so that $v_1 - v_2$ is a monotone function for small x. (34) now follows.

In analogy with (*), one might expect that the conditions (21) and (22) or (29) and (31) can be replaced by an integral condition similar to (4); for example, a condition like

$$\int_{+0} (c/\phi(x) - 1/x)\,dx = +\infty. \tag{37}$$

However, the solutions of the differential equation

$$(1/x - 1/x \log x)^{-1} y' = y + x$$

are

$$y = \tfrac{1}{2} x \log x - x + Kx/\log x,$$

where K is an arbitrary integration constant; so that it is false that (37) is sufficient for the existence of a limit of the ratio $y(x)/x$. But (37) may be used to improve Theorem 4 as follows:

THEOREM 5. *Let $\phi(x)$, $f(x, y)$ satisfy the conditions of Theorem* 4.

In addition, let (37) *be satisfied. If* $y = y_1(x)$, $y = y_2(x)$ *are any two solutions of* (2) *satisfying* (8), *then*

$$\lim_{x \to +0} (y_1(x) - y_2(x))/x = 0. \tag{38}$$

Proof. Let y_1, y_2, v_1, v_2 have the same meaning as in the proof of Theorem 4, so that (36) holds. Consider the differential equation

$$u' = u[1/\phi(x) - 1/x].$$

It is clear from (37) that for any solution

$$u = u(x) = (\text{const.}) \exp\left(-\int_x [1/\phi(t) - 1/t]dt\right),$$

$$u(x) \to 0, \; x \to +0. \tag{39}$$

Now if $v_1(x) - v_2(x)$ and $u(x)$ have a common value for some x, then, by (36), for smaller (positive) x,

$$v_1(x) - v_2(x) \leqq u(x).$$

The relation (39) and the fact that $v_1(x) - v_2(x) \geqq 0$ imply (38).

This completes the proof of Theorem 5.

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LINEAR GRAPHS OF DEGREE ≤ 6 AND THEIR GROUPS.*

By I. N. KAGNO.

1. Introduction. Let G be a connected linear graph having no simple loops (i. e. no arcs with coincident endpoints, or pairs of arcs which form a circuit) and each vertex of which is of degree ≥ 3. Any one-one, continuous map of G into itself effects a permutation among some or all of its vertices. If we denote the vertices of G by $a_1, \cdots, a_n$ then, corresponding to each map T of G into itself, there is a substitution τ on the letters $a_1, \cdots, a_n$. We shall say that G is mapped into itself by the substitution τ. Thus, if Γ denotes the group of all possible maps of G into itself, the set of corresponding substitutions constitute a substitution group $\mathfrak{G}$. We shall say that a given graph G *has the group* $\mathfrak{G}$, or that $\mathfrak{G}$ *is the group of* G, if corresponding to every map of G into itself there is a substitution of $\mathfrak{G}$, and each substitution of $\mathfrak{G}$ represents a map of G into itself. We shall also say that $\mathfrak{G}$ *has the graph* G, or that G *is the graph of* $\mathfrak{G}$.

In this paper we seek to determine whether a given graph of Degree ≤ 6 has a non-identical group (where by the Degree of a graph we mean the number of its vertices; to avoid confusion between the words *Degree* as applied to graphs and *degree* as applied to vertices, we shall use a capital D in the former case and a lower case d in the latter) and whether a given group of degree ≤ 6 has a graph.[1] We shall show that every graph of Degree ≤ 6 has a non-identical group. The converse of this is not true, for as we shall show, there exist groups of any degree > 2 which do not have graphs. We shall also give an example of a graph that does not have any non-identical group. Thus in general, not every graph has a non-identical group, nor does every group have a graph. We shall find the group for each graph of Degree ≤ 6, and examine each group of degree ≤ 6 to determine whether it has a graph.

* Received January 4, 1946; Revised May 6, 1946.

[1] The second question is a special case of the more general question proposed by König in his book "Theorie der endlichen und unendlichen Graphen," Leipzig (1936): "When can a given abstract group be set up as the group of a graph, and if possible how can the graph be constructed?" The graphs referred to by König may also contain simple loops.

I.

2. Some theorems on graphs of any Degree.

Definition. Let I_b^a be a number defined for a graph G as follows; if G contains the arc $\widehat{ab}$ then $I_b^a = I_a^b = 1$. If G does not contain the arc $\widehat{ab}$ then $I_b^a = I_a^b = 0$. I_b^a is called the *adjacency number* of the pair of vertices a, b. If G has the property that G contains the arc $\widehat{rs}$ if and only if it contains the arc $\widehat{pq}$, then we have $I_q^p = I_s^r$. If G is mapped into itself by a substitution τ, then an adjacent pair of vertices must be carried into a pair that is adjacent, and a non adjacent pair of vertices cannot be carried into an adjacent pair. In other words, τ is arc-preserving. Hence if a, b, p, q are vertices of G such that $\tau(a) = p$, $\tau(b) = q$, then $I_b^a = I_q^p$. Similarly if τ leaves a fixed and $\tau(b) = q$ then $I_b^a = I_q^a$.

THEOREM 1. *Let τ be a substitution on the vertices of a graph G such that $I_q^p = I_{\tau(q)}^{\tau(p)}$ for every pair of vertices p, q, of G. Then there exists a one-one continuous mapping σ of G into itself such that $\sigma(x) = \tau(x)$ for every vertex x of G. We shall say that τ maps G into itself.*

Proof.[2] Let $\widehat{pq}$ be any arc of G. Then, by hypothesis, G contains $\widehat{p'q'}$, where $p' = \tau(p)$, $q' = \tau(q)$. If p'', q'' are vertices such that $p'' = \tau^{-1}(p)$, $q'' = \tau^{-1}(q)$ then, by hypothesis, $I_{q'',p''} = I_{\tau(q'')}^{\tau(p'')} = I_q^p$; hence G also contains $\widehat{p''p''}$.

Let σ be a single-valued transformation of G into itself defined as follows: Suppose that P is any point of G. (a) If P is a vertex then we set $\sigma(P) = \tau(P)$. (b) If P is an interior point of an arc $\widehat{pq}$, let T_{pq} be a continuous one-one transformation of the closed arc $\widehat{pq}$ onto the closed arc $\widehat{p'q'}$, such that $T_{pq}(p) = p'$, $T_{pq}(q) = q'$. Let $\sigma(P) = T_{pq}(P)$.

Since T_{pq} is defined and continuous on the closure of $\widehat{pq}$, and $\sigma(p) = \tau(p) = p' = T_{pq}(p)$, $\sigma(q) = \tau(q) = q' = T_{pq}(q)$, it is clear that σ is a continuous transformation which carries every point of G into a unique point of G.

Let σ^* be a single-valued transformation of G into itself defined as follows; if P is a vertex of G let $\sigma^*(P) = \tau^{-1}(P)$, and if P is an interior point of an arc $\widehat{pq}$, let $\sigma^*(P) = T_{pq}^{-1}(P)$. Then as above, σ^* is continuous and carries every point of G into a unique point of G. It is readily seen that σ^*

[2] The proof which follows, an improvement over the author's original proof, was suggested by the referee. The author wishes to thank the referee for this and other valuable suggestions in the preparation of this paper for publication.

and σ are inverses of each other. Hence σ is one-one, as well as continuous. By definition $\sigma(x) = \tau(x)$ for every vertex x. Obviously σ preserves incidence. Hence σ satisfies the conditions of the theorem.

THEOREM 2. *If a graph G of Degree α_0 has a group $\mathfrak{G}$, a necessary condition that $\mathfrak{G}$ be of degree α_0 is that if a_i is a vertex of degree δ of G, then G contains at least one other vertex of degree δ.*

Proof. Suppose that a_i is of degree δ, and that G does not contain any other vertex of degree δ. If every map T of G into itself leaves a_i fixed, then the group of G cannot be of degree α_0, contrary to hypothesis. Hence let T be a map which does not leave a_i fixed and set $T(a_i) = a_j$. Then the arcs incident with a_i are mapped into arcs incident with a_j, any two distinct arcs on the former going into two distinct arcs on the latter. If the degree of a_j is less than δ this is not possible. If the degree of a_j exceeds δ, then there are arcs on a_j which are not the image of any arcs on a_i. Since T is one-one this is impossible. Hence G must contain another vertex of degree δ.

We shall call the group of substitutions on n letters, which transforms a plane n-gon into itself the *n-gonal group.*

THEOREM 3. *The complete N-point has the group $\mathfrak{S}_n \equiv (a_1 a_2 \cdots a_n)$* all. As this theorem is well known,[3] we omit the proof.

THEOREM 4. *If a graph G is mapped into itself by the substitution $\sigma = (a_1 a_2 \cdots a_n)$ on n of its vertices $a_1, a_2, \cdots, a_n$, then G is also mapped into itself by each substitution of the n-gonal group $\mathfrak{N}$ on the letters $a_1, \cdots, a_n$.*

Proof. $\sigma = (a_1 a_2 \cdots a_n)$ leaves fixed every vertex of G distinct from the vertices a_j, $(j = 1, \cdots, n)$. Let $p \neq a_j$ be a vertex of G adjacent to a_1. Since σ leaves p fixed, $I_{a_1}{}^p = \cdots = I_{a_n}{}^p = 1$. Similarly if q is any vertex of G distinct from $a_1, \cdots, a_n$,

$$I_{a_1}{}^q = \cdots = I_{a_n}{}^q. \tag{1}$$

Suppose that $I^{a_2}_{a_1+i} = 1$, $(i = 1, \cdots, n-1)$. Then $I^{\sigma(a_1)}_{\sigma(a_1+i)} = 1$. Similarly if $I^{a_1}_{a_1+i} = 0$, then $I^{\sigma(a_1)}_{\sigma(a_1+i)} = 0$. That is $I^{a_1}_{a_1+i} = I^{\sigma(a_1)}_{\sigma(a_1+i)} = I^{a_2}_{a_2+i}$ (the additions in the subscripts being understood to be modulo n). Analogously

$$I^{a_1}_{a_1+i} = I^{a_2}_{a_2+i} = \cdots = I^{a_n}_{a_n+i}. \tag{2}$$

[3] Cf. R. Frucht, "Die Gruppe des Peterschen Graphen etc.," *Com. Math. Helv.*, vol. 9 (1937), pp. 217-223.

Since σ maps G into itself, G is also mapped into itself by each of the substitutions $\sigma^2, \sigma^3, \cdots, \sigma^{n-1}$.

To show that the remaining substitutions of $\mathfrak{N}$ map G into itself, we must consider two cases; 1.) n is odd, 2.) n is even.

Case 1. n is odd. Consider the substitution

$$\tau = (a_2 a_n)(a_3 a_{n-1}) \cdots (a_j a_{n+2-j}) \cdots (a_{(n+1)/2} a_{(n+3)/2}).$$

τ leaves fixed any pair of vertices p, q distinct from a_j, $(j = 1, \cdots, n)$, and hence any arc $\widehat{pq}$. That is, $I_q^p = I_{\tau(p)}^{\tau(q)}$. From (1), if p is any vertex distinct from a_j, then $I_{a_j}^p = I_{a_{n+2-j}}^p = I_{\tau(a_j)}^{\tau(p)}$. If a_j and a_k are distinct, let $k = j + i$. Then from (2),

$$I_{a_k}^{a_j} = I_{a_{j+i}}^{a_j} = I_{a_{(n+2-k)+i}}^{a_{(n+2-k)}} = I_{a_{n+2-j}}^{a_{n+2-k}} = I_{a_{n+2-k}}^{a_{n+2-j}} = I_{\tau(a_k)}^{\tau(a_j)}.$$

Thus for any pair of vertices u, v of G, $I_v^u = I_{\tau(v)}^{\tau(u)}$. Hence by Theorem 1 τ maps G into itself.

Now any substitution of $\mathfrak{N}$ is a product of a power of τ and a power of σ (for σ and τ are the generators of the $2n$ substitutions of $\mathfrak{N}$). Hence any such substitution maps G into itself.

Case 2. n is even. Consider the substitution

$$\tau = (a_1 a_n)(a_2 a_{n-1}) \cdots (a_j a_{n+1-j}) \cdots (a_{n/2} a_{(n+2)/2}).$$

τ leaves fixed any pair of vertices p, q distinct from a_j. That is $I_q^p = I_{\tau(q)}^{\tau(p)}$. From (1), if p is any vertex distinct from a_j, $I_{a_j}^p = I_{\tau(a_j)}^{\tau(p)}$. If a_j and a_k are distinct, then from (2)

$$I_{a_k}^{a_j} = I_{a_{(n+1-k)+i}}^{a_{(n+1-k)}} = I_{a_{n+1-j}}^{a_{n+1-k}} = I_{a_{n+1-k}}^{a_{n+1-j}} = I_{\tau(a_k)}^{\tau(a_j)}.$$

Thus for any pair of vertices u, v of G, $I_v^u = I_{\tau(v)}^{\tau(u)}$. Hence by Theorem 1, τ maps G into itself.

In an analogous manner we can show that the substitution

$$\rho = (a_2 a_n)(a_3 a_{n-1}) \cdots (a_j a_{n+2-j}) \cdots (a_{n/2} a_{(n+4)/2})$$

maps G into itself.

Since $\mathfrak{N}$ is generated by the substitutions σ, τ, and ρ,[4] any substitution of $\mathfrak{N}$ maps G into itself.

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[4] Cf. Miller, Blichfeldt and Dickson, *Theory and Applications of Finite Groups*, p. 9.

COROLLARY 1. *The cyclic group* $(abc\cdots n)$cyc *on n letters,* $(n>2)$ *does not have a graph.*

Proof. Suppose that G is the graph of the group $(abc\cdots n)$cyc. Then G is mapped into itself by the substitution $\sigma = (abc\cdots n)$. But then by the Theorem, G is mapped into itself by the substitution $\tau_1 = (a_2a_n)\cdots(a_ja_{n+2-j})\cdots(a_{(n+1)/2}a_{(n+3)/2})$ when n is odd, or $\tau_2 = (a_1a_n)\cdots(a_ja_{n-j+1})\cdots(a_{n/2}a_{(n+2)/2})$ when n is even. Since neither of these substitutions is contained in the given group this gives a contradiction.

COROLLARY 2. *The group* $(a_1a_2\cdots a_n)$cyc$(pq\cdots s)$ *does not have any graph.*

Proof. Suppose that the given group has the graph G. Then G is mapped into itself by the substitution $(a_1a_2\cdots a_n)$ on n of its vertices $a_1, a_2, \cdots, a_n$, for the given group is the direct product of the groups $\mathfrak{G}_1 \equiv (a_1a_2\cdots a_n)$cyc and $\mathfrak{G}_2 \equiv [1, (pqr\cdots s)]$. But by the Theorem, if n is odd G is then mapped into itself by the substitution $\tau_1 = (a_2a_n)\cdots(a_ja_{n+2-j})\cdots(a_{(n+1)/2}a_{(n+3)/2})$ and if n is even G is mapped into itself by the substitution $\tau_2 = (a_1a_n)\cdots(a_ja_{n-j+1})\cdots(a_{n/2}a_{(n+2)/2})$. Since neither τ_1 nor τ_2 are contained in the given group, this gives a contradiction.

THEOREM 5. *The substitution group* $(abc\cdots m)$pos *does not have any graph.*

Proof. Since the group contains only positive substitutions, it does not contain any transpositions. Furthermore since the group contains all positive substitutions on m letters, it contains in particular the substitution (abc). Now suppose that the group has the graph G. Then G is mapped into itself by the substitution (abc) on three of its vertices a, b, c. But then G is mapped into itself by all substitutions of the 3-gonal group (abc)all, (Theorem 4), and in particular by (ab), which gives a contradiction.

3. Example of a graph not having any non-identical group.

Definition. We shall describe a graph G by the symbol

$$G \equiv [rs, rt, \cdots, xy, \cdots],$$

where the couples $rs, rt, \cdots, xy, \cdots$ indicate that G contains the arcs $\widehat{rs}, \widehat{rt}, \cdots$ etc.

By the corollaries to Theorem 3, and by Theorem 4, we have shown the

existence of groups that do not have any graphs. We now give an example of a graph that does not have any non-identical group. Let M be the graph

$$M \equiv [qr, qs, qt, qv, qw, ru, rv, rw, st, su, tw, uv].$$

Let τ be any map of M into itself. We shall show that τ is the identity. Since r is the only vertex of degree 4, $\tau(r) = r$, (Theorem 1). Since q is the only vertex of degree 5, τ also leaves q fixed, and hence carries the arc $\widehat{qr}$ into itself and permutes the vertices s, t, v and w adjacent to q among themselves. Since $I_v{}^r = 1$, $I_s{}^r = 0$, τ cannot carry v into s. Since $I_v{}^r = 1$, $I_t{}^r = 0$, $\tau(v) \neq t$. Suppose that $\tau(v) = w$, then τ carries the set of vertices (q, r, u) adjacent to v into the set (q, r, t) adjacent to w. Since q and r are fixed, it follows that $\tau(u) = t$. But since $I_u{}^r = 1$, $I_t{}^r = 0$, τ cannot carry u into t. Hence $\tau(v) \neq w$, and the only remaining possibility is $\tau(v) = v$. Since τ leaves v fixed it permutes the set of vertices (q, r, u) adjacent to v among themselves, and since q and r are fixed it follows that u also is fixed. Since u is fixed τ permutes the vertices (r, v, s) adjacent to u, and since r and v are fixed, s must also be fixed. Since s is fixed τ permutes the pair of vertices (u, t) adjacent to s, and since u is fixed, it follows that t is also fixed. Hence w, the only remaining vertex is also fixed. Thus we have shown that τ leaves every vertex of M fixed, hence also the arcs of M. That is M is invariant under τ and $\tau = 1$.

We note that if a graph of Degree α_0 does not have a non-identical group of degree α_0, it does not necessarily follow that this graph does not have any non-identical group whatsoever. For example, the graph G_3 of Sec. II below is of Degree 5, and does not have any non-identical group of degree 5, but has the octic group (of degree 4).

4. Preliminary Lemmas. The proofs of many theorems of Section III are based on the following lemmas, which apply to graphs of any Degree.

LEMMA 1. *If a graph is mapped into itself by a transitive group of substitutions on n of its vertices $a_1, \cdots, a_n$, then these vertices are all of the same degree.*

Proof. Suppose that a_i and a_j are of different degrees. Let σ be the substitution which carries a_i into a_j. Then vertices adjacent to a_i are carried by σ into vertices adjacent to a_j, each arc $\widehat{a_i p}$ incident with a_i being carried into an arc $\widehat{a_j q}$ incident with a_j, and each arc $\widehat{a_j q'}$ incident with a_j being the image of an arc incident with a_i. Since σ is one-one, this is impossible if a_i and a_j are of different degrees.

COROLLARY. *The group* $\mathfrak{G} \equiv (abcd)_4$ *does not have any graph.*

Proof. Suppose that $\mathfrak{G}$ has the graph G. $\mathfrak{G}$ consists of the substitutions 1, $(ab)(cd)$, $(ac)(bd)$ and $(ad)(bc)$. Since $\mathfrak{G}$ is transitive, a, b, c, d must all have the same degree in G. Also if p is any vertex of G distinct from a, b, c, d, then every substitution of $\mathfrak{G}$ leaves p fixed and $I_a{}^p = I_b{}^p = I_c{}^p = I_d{}^p$. Let $K = G -$ [the set of arcs of G not joining pairs of the vertices a, b, c, d]. Then in K, a, b, c, d are also all of the same degree.

If a, b, c, d are each of degree 1 in K, then K consists of the two arcs $\widehat{ab}, \widehat{cd}$ or of the two arcs $\widehat{ac}, \widehat{bd}$, or of the two arcs $\widehat{ad}, \widehat{bc}$. But then G is mapped into itself by the substitutions (ab), (ac) or (ad) respectively. Since $\mathfrak{G}$ contains neither of these substitutions, this gives a contradiction.

If a, b, c, d are each of degree 2 in K, then K consists of a circuit, say $\overset{\frown}{abcd}$. But then G is mapped into itself by the substitution $(abcd)$, which is not contained in $\mathfrak{G}$, giving a contradiction.

Finally, if a, b, c, d are each of degree 3 in K, then K is a complete 4-point. But then G is mapped into itself by the substitution $(abcd)$, again giving a contradiction.

LEMMA 2. *If a graph* G *is mapped into itself by the substitution* $\sigma = (abc)(def)$ *on six of its vertices* a, b, c, d, e, f, *then* G *is mapped into itself by every substitution of some one of the groups* $\mathfrak{G} \equiv (abc \cdot def)$all, $\mathfrak{G}' \equiv (abc \cdot fde)$all, *or* $\mathfrak{G}'' \equiv (abc \cdot efd)$all.[5]

Proof. σ leaves fixed any vertex p distinct from a, b, c, d, e, f. Hence

$$I_a{}^p = I_b{}^p = I_c{}^p \text{ and } I_d{}^p = I_e{}^p = I_f{}^p. \tag{3}$$

Also $I_d{}^a = I_e{}^b = I_f{}^c \equiv I_1$, $I_d{}^b = I_e{}^c = I_f{}^a \equiv I_2$, $I_d{}^c = I_e{}^a = I_f{}^b \equiv I_3$ and

$$I_b{}^a = I_c{}^b = I_a{}^c, \qquad I_e{}^d = I_f{}^e = I_d{}^f. \tag{4}$$

Case 1.

$$I_2 = I_3 \neq I_1 \tag{5}$$

Consider the substitution $\tau = (ab)(de)$. If pq is any arc not incident with $a, b, d,$ or e, then τ leaves p, q, and $\widehat{pq}$ fixed and hence $I_q{}^p = I_{\tau(q)}{}^{\tau(p)}$. From this and examination of relations (3), (4) and (5) we see that for any pair of

[5] The substitutions of $\mathfrak{G}$ are given by G. A. Miller, "Memoir on the Substitution Groups whose degree does not exceed eight," *American Journal of Mathematics*, vol. 21 (1899), pp. 287-337, $\mathfrak{G}' \equiv [1, (abc)(def), (acb)(dfe), (ab)(df), (bc)(de), (ac)(ef)]$, $\mathfrak{G}'' \equiv [1, (abc)(def), (acb)(dfe), (ab)(ef), (bc)(fd), (ac)(de)]$.

vertices, u, v of G, $I_v{}^u = I_{\tau(v)}{}^{\tau(u)}$. Hence by Theorem 1, τ maps G into itself. Since $\mathfrak{G}$ can be generated by σ and τ, any substitutes of $\mathfrak{G}$ maps G into itself.

Case 2. $I_1 = I_3 \neq I_2$. By analogy with case 1 we can show that G is mapped into itself by the substitution $\nu = (ac)(ef)$. Since $\mathfrak{G}'$ can be generated by σ and ν, any substitution of $\mathfrak{G}'$ maps G into itself.

Case 3. $I_1 = I_2 \neq I_3$. By analogy with case 1 we can show that G is mapped into itself by the substitution $\rho = (bc)(df)$. Since $\mathfrak{G}''$ can be generated by σ and ρ, any substitution of $\mathfrak{G}''$ maps G into itself.

Case 4. $I_1 = I_2 = I_3$. As in case 1, we can show that G is mapped into itself by the substitution τ, and that any substitution of $\mathfrak{G}$ maps G into itself.

LEMMA 3. *If a graph G is mapped into itself by the substitution $\tau = (abcd)(ef)$ on six of its vertices a, b, c, d, e, f, then G is mapped into itself by every substitution of the group $\mathfrak{G} \equiv \{(abcd)_8 \mathrm{com}(ef)\}\mathrm{dim}$.*

Proof. Suppose that G is mapped into itself by τ. If q is a vertex distinct from $a, \cdots, f$, then $I_a{}^q = I_b{}^q = I_c{}^q = I_d{}^q$ and $I_e{}^q = I_f{}^q$. Also $I_b{}^a = I_c{}^b = I_d{}^c = I_a{}^d$, $I_e{}^a = I_e{}^c$, $I_f{}^a = I_f{}^c$, $I_f{}^b = I_e{}^c$, $I_e{}^c = I_f{}^d$, $I_f{}^d = I_e{}^a$.

Consider the substitution $\sigma = (ac)$. If $\widehat{pq}$ is an arc not incident with a or c then σ leaves p, q, and $\widehat{pq}$ fixed, hence $I_q{}^p = I_{\sigma(q)}^{\sigma(p)}$. If r is any vertex distinct from a or c, from the preceding paragraph we see that $I_a{}^r = I_c{}^r = I_{\sigma(a)}^{\sigma(r)}$ and $I_c{}^r = I_{\sigma(c)}^{\sigma(r)}$. Also $I_c{}^a = I_a{}^c$, hence $I_v{}^u = I_{\sigma(v)}^{\sigma(u)}$ for any pair of vertices u and v. That is, σ maps G into itself (Theorem 1). Since $\mathfrak{G}$ is generated by σ and τ, every substitution of $\mathfrak{G}$ maps G into itself.

LEMMA 4. *If a graph G is mapped into itself by the substitutions $\tau = (ab)(cd)$ and $\sigma = (cd)(ef)$ on six of its vertices a, b, c, d, e, f, then G is mapped into itself by every substitution of the group $\mathfrak{G} \equiv (ab)(cd)(ef)$.*

Proof. G is mapped into itself by τ. Let q be a vertex distinct from a, b, c, d. Then $I_a{}^q = I_b{}^q$, $I_c{}^q = I_d{}^q$, and $I_c{}^a = I_d{}^b$, $I_c{}^b = I_d{}^a$.

G is also mapped into itself by σ, hence $I_c{}^a = I_d{}^a$ and $I_c{}^b = I_d{}^b$. Combining these adjacency relations with those of the preceding paragraph, we have $I_c{}^a = I_d{}^a = I_c{}^b = I_d{}^b$.

Consider the substitution $\mu = (ab)$. By discussion similar to that in the proof of Lemma 3, we can show that μ maps G into itself. Since $\mathfrak{G}$ can be generated by τ, σ, and μ, every substitution of $\mathfrak{G}$ maps G into itself.

II. Groups of the graphs of Degree ≤ 6.[9]

The proofs of the theorems in this section, and Section III, are repetitious in form, therefore to keep the length of this paper within reasonable bounds we shall for the most part delete these proofs, retaining only a few to illustrate the methods used, or those that differ materially from the general method. Except where otherwise stated, the omitted proofs are similar in form to those of Theorems 2.1 and/or 2.2 below.

Definition. We shall say that a graph is *admissible* if it is connected and contains no simple loops and no vertex of degree < 3.

1. Graphs of Degree 4. The only admissible graph is the complete 4-point N_4. N_4 has the group $\mathfrak{S}_4 \equiv (abcd)\text{all}$, (Theorem 3).

2. Graphs of Degree 5. There are three admissible graphs,

$$G_1 \equiv N_5, \text{ the complete 5-point.}$$
$$G_2 \equiv [ab, ac, ad, ae, bc, bd, be, cd, ce]$$
$$G_3 \equiv [ac, ad, ae, bc, bd, be, ce, de].$$

The graph G_1 has the group $\mathfrak{G}_8 \equiv (abcde)\text{all}$, (Theorem 3).

THEOREM 2.1. *G_2 has the group $\mathfrak{G}_5 \equiv (abc)\text{all}(de)$.*

Proof. It is readily verified that any substitution of $\mathfrak{G}_5$ maps G_2 into itself. Suppose that G_2 is mapped into itself by a substitution τ. Since d and e are the only vertices of degree 3, τ either leaves this pair fixed or interchanges them.

Case 1. τ leaves d and e fixed. Then $\tau = \left(\begin{smallmatrix} abc \\ \alpha\beta\gamma \end{smallmatrix}\right)$, where α, β, γ is some permutation of the letters a, b, c. Now $\mathfrak{G}_5$ contains a substitution σ such that $\sigma^{-1} = \left(\begin{smallmatrix} \alpha\beta\gamma \\ abc \end{smallmatrix}\right)$. Hence $\tau\sigma^{-1} = 1$ and $\tau = \sigma \in \mathfrak{G}_5$.

Case 2. τ interchanges d and e. Then $\tau = \left(\begin{smallmatrix} abc \\ \alpha\beta\gamma \end{smallmatrix}\right)(de)$. Since $\mathfrak{G}_5$ contains the substitution $\mu = (de)$, $\tau\sigma^{-1}\mu = 1$. Hence $\tau = \mu\sigma \in \mathfrak{G}_5$.

THEOREM 2.2. *G_3 has the octic group on the four letters a, b, c, d. G_3 does not have a group of degree* 5.

Proof. Since G_3 has only one vertex e of degree 4, it cannot have a group of degree 5 (Theorem 1).

It can readily be verified that G_3 is mapped into itself by the substitution $\sigma = (abcd)$. Hence by Theorem 4, G_3 is mapped into itself by every substitution of the octic group $\mathfrak{R}_4$ ($\equiv \mathfrak{F}_5$).

Suppose that G_3 is mapped into itself by the substitution τ. Since e is the only vertex of degree 4, τ leaves e fixed.

Case 1. $\tau(a) = a$. Then since $I_c{}^a = 1$, $I_d{}^a = 1$, either $\tau(c) = c$, $\tau(d) = d$, in which case $\tau(b) = b$, $\tau = 1$, or else $\tau(c) = d$, $\tau(d) = c$, in which case $\tau(b) = b$ and $\tau = (cd) \in \mathfrak{N}_4$.

Case 2. $\tau(a) = b$. Then since $I_c{}^a = I_d{}^a = I_c{}^b = 1$, either $\tau(c) = d$, $\tau(d) = c$, and $\tau = (ab)(cd) \in \mathfrak{N}_4$, or $\tau(c) = c$, $\tau(d) = d$ and $\tau = (ab) \in \mathfrak{N}_4$.

Case 3. $\tau(a) = c$ or d. A similar discussion shows that $\tau \in \mathfrak{N}_4$.

3. Graphs of Degree 6. There are eighteen admissible graphs,

$H_1 \equiv N_6$, the complete 6-point.
$H_2 \equiv [ab, ac, ad, ae, af, bc, bd, be, bf, cd, ce, cf, de, df]$
$H_3 \equiv [ab, ac, ad, ae, af, bc, bd, be, bf, cd, ce, cf, de]$
$H_4 \equiv [ac, ad, ae, af, bc, bd, be, bf, ce, cf, de, df, ef]$
$H_5 \equiv [ab, ad, ae, af, bc, bd, be, bf, ce, cf, de, df]$
$H_6 \equiv [ac, ad, ae, af, bc, bd, be, bf, cd, cf, df, ef]$
$H_7 \equiv [ab, ac, ae, af, bd, be, bf, ce, cf, de, df, ef]$
$H_8 \equiv [ab, ad, ae, af, bc, be, bf, ce, cf, de, df]$
$H_9 \equiv [ab, ac, ae, af, bd, be, bf, cd, cf, df, ef]$
$H_{10} \equiv [ad, ae, af, bd, be, bf, cd, ce, cf, df, ef]$
$H_{11} \equiv [ab, ac, ad, ae, bc, bd, bf, cd, ce, df, ef]$
$H_{12} \equiv [ad, ae, af, bc, be, bf, ce, cf, de, df, ef]$
$H_{13} \equiv [ad, ae, af, bc, be, bf, ce, cf, de, df]$
$H_{14} \equiv [ad, ae, af, bc, be, bf, cd, ce, df, ef]$
$H_{15} \equiv [ab, ae, af, bc, bf, cd, cf, de, df, ef]$
$H_{16} \equiv [ad, ae, af, bd, be, bf, cd, ce, cf, de]$
$H_{17} \equiv [ac, ad, ae, bd, be, bf, ce, cf, df]$
$H_{18} \equiv [ad, ae, af, bd, be, bf, cd, ce, cf]$

By Theorem 3, the graph H_1 has the group $\mathfrak{H}_{37} \equiv (abcdef)\text{all}$.

THEOREM 3.1. *H_2 has the group $\mathfrak{H}_{31} \equiv (abcd)\text{all}(ef)$.*

THEOREM 3.2. *H_3 has the group $\mathfrak{G}_5 \equiv (abc)\text{all}(de)$ on five of its vertices a, b, c, d, e. H_3 does not have a group of degree 6.*

THEOREM 3.3. *H_4 has the group $\mathfrak{H}_{19} \equiv (abcd)_8(ef)$.*

THEOREM 3.4. *H_5 has the group $\mathfrak{H}_{19} \equiv (abcd)_8(ef)$.*

$\mathfrak{H}_{32} \equiv (abcdef)_{48}$

H_5 is homeomorphic with the graph formed by the edges and vertices of the regular octahedron.

THEOREM 3.5. *H_6 has the group $\mathfrak{F}_2 \equiv (ab)(cd)$, on four letters. H_6 does not have a group of degree 6.*[6]

THEOREM 3.6. *H_7 has the group $\mathfrak{H}_3 \equiv (ab \cdot cd)(ef)$.*

THEOREM 3.7. *H_8 has the group $\mathfrak{H}_3 \equiv (ab \cdot cd)(ef)$.*

THEOREM 3.8. *H_9 has the group $\mathfrak{F}_1 \equiv (ab \cdot cd)$ on four letters. H_9 does not have any group of degree 6.*[7]

THEOREM 3.9. *H_{10} has the group $\mathfrak{G}_5 \equiv (abc)\mathrm{all}(de)$, on five letters. H_{10} does not have a group of degree 6.*[8]

THEOREM 3.10. *H_{11} has the group $\mathfrak{H}_{13} \equiv \{(abcd)_8 \mathrm{com}(ef)\}\mathrm{dim}$.*

THEOREM 3.11. *H_{12} has the group $\mathfrak{H}_{19} \equiv (abcd)_8(ef)$.*

THEOREM 3.12. *H_{13} has the group $\mathfrak{H}_{19} \equiv (abcd)_8(ef)$.*

THEOREM 3.13. *H_{14} has the group $\mathfrak{H}_1 \equiv (ab \cdot cd \cdot ef)$.*

THEOREM 3.14. *H_{15} has the group $\mathfrak{H}_{12} \equiv (abcde)_{10}$ on five letters. H_{15} does not have a group of degree 6.*

Proof. Since H_{15} has but one vertex f of degree 5, by Theorem 2 its group cannot be of degree 6. Since f is adjacent to each of the other vertices, any map of K into itself, where $K \equiv H_{15} - [f +$ the set of arcs incident with $f]$, is also a map of H_{15} into itself. That is, the group of K is also the group of H_{15}. Since K is the circuit $\overset{\frown}{abcde}$, K has the group $(abcde)_{10}$.

THEOREM 3.15. *H_{16} has the group $\mathfrak{G}_5 \equiv (abc)\mathrm{all}(de)$, on five letters. H_{16} does not have a group of degree 6.*

THEOREM 3.16. *H_{17} has the group $\mathfrak{H}_{17} \equiv (abcdef)_{12}$.*

Proof. It is readily verified that H_{17} is mapped into itself by every substitution of $\mathfrak{H}_{17}$. Suppose that H_{17} is mapped into itself by a non-identical substitution τ. Then some vertex of H_{17} is not left fixed by τ, say $\tau(a) \neq a$. Let $\alpha = \tau(a)$. Since $\mathfrak{H}_{17}$ is transitive it contains a substitution σ such that

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[6] The group $\mathfrak{F}_2$ is simply isomorphic with $\mathfrak{H}_3 \equiv (ab \cdot cd)(ef)$ of degree 6; *cf.* Miller, *loc. cit.*

[7] $\mathfrak{F}_1$ is simply isomorphic with $\mathfrak{H}_1 \equiv (ab \cdot cd \cdot ef)$ of degree 6, Miller, *loc. cit.*

[8] $\mathfrak{G}_5$ is simply isomorphic with $\mathfrak{H}_{17} \equiv (abcdef)_{12}$ of degree 6, *cf.* Miller, *loc. cit.*

$\sigma(\alpha) = a$. Then $\tau' = \tau\sigma$ maps H_{17} into itself and leaves a fixed. Since $I_c{}^a = 1$, $I_b{}^a = 0$, $I_f{}^a = 0$ τ' cannot carry c into b or f. Similarly τ' cannot carry d or e into b or f. That is, τ' permutes the vertices c, d, e among themselves, and hence either leaves the pair b, f fixed or interchanges them.

Case 1. τ' leaves fixed the pair b, f. Since $I_d{}^f = 1$, $I_e{}^f = 0$, $\tau'(d) \neq e$. Since $I_d{}^b = 1$, $I_c{}^b = 0$, $\tau'(d) \neq c$. Hence τ' leaves d fixed. Since $I_c{}^f = 1$, $I_e{}^f = 0$, $\tau'(c) \neq e$. Hence τ' leaves c fixed, and therefore the remaining vertex e is left fixed. That is $\tau' = \tau\sigma = 1$. Hence $\tau = \sigma^{-1} \varepsilon \mathfrak{H}_{17}$.

Case 2. τ' interchanges b and f. Let $\mu = (bf)(ce) \varepsilon \mathfrak{H}_{17}$. Then $\tau'' = \tau'\mu$ maps H_{17} into itself, leaving fixed the vertices a, b, f and permuting the vertices c, d, e among themselves. As in case 1, we can show that $\tau'' = \tau\sigma\mu = 1$. That is, $\tau = \mu^{-1}\sigma^{-1} \varepsilon \mathfrak{H}_{17}$.

THEOREM 3.17. *H_{18} has the group $\mathfrak{H}_{34} \equiv (abcdef)_{72}$.*

The proof is similar to that of Theorem 3.16.

The graph H_{18} is of special interest, since it is one of the two irreducible non-planar graphs (the other being the complete 5-point).

Since we have shown the existence of a non-identical group for each admissible graph of Degree ≤ 6, we have proved the following theorem:

THEOREM 6. *Any graph of Degree ≤ 6, containing no simple loops, and no vertices of degree < 3, has a non-identical group.*

III. Graphs of the substitution groups of degree ≤ 6.

A. Groups of degree 2 and 3.[9] The groups are,

$$\mathfrak{E}_1 \equiv (ab), \qquad \mathfrak{E}_2 \equiv (abc)\text{cyc}, \qquad \mathfrak{E}_3 \equiv (abc)\text{all}.$$

THEOREM A. 1. *The group $\mathfrak{E}_1$ has the graph*

$$E_1 \equiv [ab] + M + [\textit{the set of arcs joining } a \textit{ and } b \textit{ to each vertex of } M],$$

where M is the graph of Section I, 3.

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[9] A complete list of the substitutions of $\mathfrak{H}_{29}$ and $\mathfrak{H}_{32}$ are given by Cole, "List of substitution groups of Nine Letters," *Quarterly Journal of Mathematics*, vol. 26 (1892), p. 372 *et seq.*

A complete list of the substitutions of $\mathfrak{H}_{18}$, $\mathfrak{H}_{24}$, $\mathfrak{H}_{27}$, $\mathfrak{H}_{28}$, $\mathfrak{H}_{31}$, $\mathfrak{H}_{33}$, $\mathfrak{H}_{34}$, $\mathfrak{H}_{35}$, are given by Cayley, "On substitution groups . . . etc.," *Quarterly Jounral of Mathematics*, vol. 25 (1891), pp. 71-88.

A complete list of the substitutions of all the other groups are given by Miller, *loc. cit.*

The proof is similar in form to that of Theorem B. 1 below.

The group $\mathfrak{E}_2$ does not have any graph ((Theorem 4, Corollary 1).

THEOREM A. 2. *The group $\mathfrak{E}_3$ has the graph*

$E_3 \equiv [ab, ac, bc] + M +$ [*the set of arcs joining a, b, c to each vertex of M*], *where M is the graph of Section* I, 3.

The proof is similar in form to that of the first paragraph of the proof of Theorem B. 1 and the proof of Theorem 2. 2.

B. Groups of degree 4.[9] There are seven groups of degree 4. They are, in sequence of increasing order,

$$\mathfrak{F}_1 \equiv (ab \cdot cd), \quad \mathfrak{F}_2 \equiv (ab)(cd), \quad \mathfrak{F}_3 \equiv (abcd)_4, \quad \mathfrak{F}_4 \equiv (abcd)\text{cyc},$$
$$\mathfrak{F}_5 \equiv (abcd)_8, \quad \mathfrak{F}_6 \equiv (abcd)\text{pos}, \quad \mathfrak{F}_7 \equiv (abcd)\text{all}.$$

The group $\mathfrak{F}_1$ has the graph H_9, (Theorem 3. 8).

THEOREM B. 1. *The group $\mathfrak{F}_2$ has the graph*

$$F_2 \equiv [ac, ad, bc, bd, cd] + M$$

$+$ [*the set of arcs joining each of the vertices a, b, c, d to each vertex of M*] *where M is the graph of Section* I, 3.

Proof. It is readily verified that the substitutions of $\mathfrak{F}_2$ map F_2 into itself. Suppose that F_2 is mapped into itself by a substitution τ. Since c and d are the only vertices of degree 10, τ either leaves this pair fixed or interchanges them. Since r is the only vertex of degree 8, τ leaves r fixed. By a proof similar to that used to show that M has no non-identical group, we can show that τ leaves each of the vertices s, t, u, v, w fixed. The vertices a, b, q are each of degree 9. Since $I_u^a = 1$, $I_u^q = 0$, τ cannot carry a into q. Similarly $\tau(b) \neq q$. Hence τ either leaves the pair a, b fixed or interchanges them. Consequently q is left fixed. That is τ leaves every vertex of M fixed, and thus also every arc of M.

Case 1. τ leaves a and b fixed. Then if τ leaves c and d fixed, $\tau = 1$; while if τ interchanges c and d, then $\tau(\widehat{cq}) = \widehat{dq}, \cdots, \tau(\widehat{cw}) = \widehat{dw}$, $\tau(\widehat{dq}) = \widehat{cq}, \cdots, \tau(\widehat{dw}) = \widehat{cw}$. That is $\tau = (cd) \in \mathfrak{F}_2$.

Case 2. τ interchanges a and b. Then if τ leaves c and d fixed, $\tau = (ab) \in \mathfrak{F}_2$; while if τ interchanges c and d, $\tau = (ab)(cd) \in \mathfrak{F}_2$.

The group $\mathfrak{F}_3$ does not have any graph (Lemma 1, Corollary).
The group $\mathfrak{F}_4$ does not have any graph (Theorem 4, Corollary 1).
The group $\mathfrak{F}_5$ has the graph G_3 (Theorem 2.2).
The group $\mathfrak{F}_6$ does not have any graph (Theorem 5).
The group $\mathfrak{F}_7$ has the complete 4-point for its graph (Theorem 3).

C. Groups of degree 5.[9] There are eight groups of degree 5. They are, in sequence of increasing order,

$$\mathfrak{G}_1 \equiv (abcde)\text{cyc}, \quad \mathfrak{G}_2 \equiv (abc)\text{cyc}(de), \quad \mathfrak{G}_3 \equiv \{(abc)\text{all}(de)\}\text{pos}$$
$$\mathfrak{G}_4 \equiv (abcde)_{10}, \quad \mathfrak{G}_5 \equiv (abc)\text{all}(de), \quad \mathfrak{G}_6 \equiv (abcde)_{20},$$
$$\mathfrak{G}_7 \equiv (abcde)\text{pos}, \quad \mathfrak{G}_8 \equiv (abcde)\text{all}.$$

The group $\mathfrak{G}_1$ does not have any graph (Theorem 4, Corollary 1).
The group $\mathfrak{G}_2$ does not have any graph (Theorem 4, Corollary 2).

THEOREM C. 1. *$\mathfrak{G}_3$ does not have any graph.*

Proof. Suppose that $\mathfrak{G}_3$ has a graph G. Then G is mapped into itself by the substitution (abc) on three of its vertices a, b, c. But then by Theorem 4, G is mapped into itself by the substitution (ab). Since G_3 does not contain (ab) this gives a contradiction.

THEOREM C. 2. *$\mathfrak{G}_4$ has the graph*

$$G_4 \equiv [ab, ae, bc, cd, de] + M$$

$+$ [*the set of arcs joining each of the vertices a, b, c, d, e to each vertex of M*], *where M is the graph of Section* I, 3.

The proof is similar to that of Theorem B. 1.

The group $\mathfrak{G}_5$ has the graph G_2 (Theorem 2. 1), and also the graphs H_3, H_{10} and H_{16} (Theorems 3. 2, 3. 9, 3. 15).

THEOREM C. 3. *$\mathfrak{G}_6$ does not have any graph.*

The proof is similar to that of Theorem C. 1.

The group $\mathfrak{G}_7$ does not have any graph (Theorem 5).
The group $\mathfrak{G}_8$ has the complete 5-point for its graph (Theorem 3).

D. Groups of degree 6.[9] There are thirty-seven groups of degree 6. Arranged in sequence of increasing order they are,

$\mathfrak{H}_1 \equiv (ab \cdot cd \cdot ef)$; $\mathfrak{H}_2 \equiv (abc \cdot def)\text{cyc}$; $\mathfrak{H}_3 \equiv (ab \cdot cd)(ef)$;
$\mathfrak{H}_4 \equiv \{(ab)(cd)(ef)\}\text{pos}$; $\mathfrak{H}_5 \equiv \{(abcd)_4(ef)\}\text{dim}$; $\mathfrak{H}_6 \equiv \{(abcd)\text{cyc}(ef)\}\text{pos}$;
$\mathfrak{H}_7 \equiv (abcdef)\text{cyc}$; $\mathfrak{H}_8 \equiv (abc \cdot def)\text{all}$; $\mathfrak{H}_9 \equiv (abcdef)_6$;
$\mathfrak{H}_{10} \equiv (ab)(cd)(ef)$; $\mathfrak{H}_{11} \equiv (abcd)\text{cyc}(ef)$; $\mathfrak{H}_{12} \equiv (abcd)_4(ef)$;
$\mathfrak{H}_{13} \equiv \{(abcd)_8\text{com}(ef)\}\text{dim}$; $\mathfrak{H}_{14} \equiv \{(abcd)_8\text{cyc}(ef)\}\text{dim}$; $\mathfrak{H}_{15} \equiv \{(abcd)_8\text{pos}(ef)\}\text{dim}$;
$\mathfrak{H}_{16} \equiv (abc)\text{cyc}(def)\text{cyc}$; $\mathfrak{H}_{17} \equiv (abcdef)_{12}$; $\mathfrak{H}_{18} \equiv (abcdef)_{12_2}$;
$\mathfrak{H}_{19} \equiv (abcd)_8(ef)$; $\mathfrak{H}_{20} \equiv (abc)\text{all}(def)\text{cyc}$; $\mathfrak{H}_{21} \equiv \{(abc)\text{all}(def)\text{all}\}\text{pos}$;
$\mathfrak{H}_{22} \equiv (abcdef)_{18}$; $\mathfrak{H}_{23} \equiv (abcd)\text{pos}(ef)$; $\mathfrak{H}_{24} \equiv \{(abcd)\text{all}(ef)\}\text{pos}$;
$\mathfrak{H}_{25} \equiv (\pm\, abcdef)_{24}$; $\mathfrak{H}_{26} \equiv (+\, abcdef)_{24}$; $\mathfrak{H}_{27} \equiv (abcdef)_{24_5}$;
$\mathfrak{H}_{28} \equiv (abc)\text{all}(def)\text{all}$; $\mathfrak{H}_{29} \equiv (abcdef)_{36}$; $\mathfrak{H}_{30} \equiv (abcdef)_{36_3}$;
$\mathfrak{H}_{31} \equiv (abcd)\text{all}(ef)$; $\mathfrak{H}_{32} \equiv (abcdef)_{48}$; $\mathfrak{H}_{33} \equiv (abcdef)_{60}$;
$\mathfrak{H}_{34} \equiv (abcdef)_{72}$; $\mathfrak{H}_{35} \equiv (abcdef)_{120}$; $\mathfrak{H}_{36} \equiv (abcdef)\text{pos}$;
$\mathfrak{H}_{37} \equiv (abcdef)\text{all}$.

The group $\mathfrak{H}_1$ has the graph H_{14} (Theorem 3. 13).

THEOREM D. 1. *The groups $\mathfrak{H}_2$ and $\mathfrak{H}_9$ have no graphs.*

The proofs, based on Lemma 2, are similar to that of Theorem C. 1.

The group $\mathfrak{H}_3$ has the graph H_7 (Theorem 3. 6).

THEOREM D. 2. *The groups $\mathfrak{H}_4$ and $\mathfrak{H}_{18}$ have no graphs.*

The proofs, based on Lemma 4, are similar to that of Theorem C. 1.

THEOREM D. 3. *The group $\mathfrak{H}_5$ has the graph*

$$R \equiv [ab, ac, ae, ag, bd, be, bg, cd, cf, cg, df, dg, eg, fg].$$

THEOREM D. 4. *The groups $\mathfrak{H}_6$, $\mathfrak{H}_{15}$, and $\mathfrak{H}_{26}$ have no graphs.*

The proofs, based on Lemma 3, are similar to that of Theorem C. 1.

The group $\mathfrak{H}_7$ does not have any graph (Theorem 4, Corollary 1).

THEOREM D. 5. *The group $\mathfrak{H}_8$ has the graph*

$$L \equiv [ab, ac, ae, af, ag, bc, bd, bf, bg, cd, ce, cg, dg, eg, fg].$$

THEOREM D. 6. *The group $\mathfrak{H}_{10}$ has the graph*

$$N \equiv [ac, ad, ag, bc, bd, bg, ce, cf, cg, de, df, dg, ef].$$

THEOREM D. 7. *The groups $\mathfrak{H}_{11}$, $\mathfrak{H}_{14}$, $\mathfrak{H}_{25}$, and $\mathfrak{H}_{35}$ have no graphs.*

The proofs, based on Theorem 4, are similar to that of Theorem C. 1.

THEOREM D. 8. *$\mathfrak{H}_{12}$ does not have any graph.*

Proof. $\mathfrak{H}_{12}$ contains as subgroup the group $\mathfrak{F}_3 \equiv (abcd)_4$. Suppose that $\mathfrak{H}_{12}$ has a graph H. Then H is mapped into itself by all the substitutions of $\mathfrak{F}_3$.

But then, as is shown in the proof of the corollary to Lemma 1, H is also mapped into itself by one of the substitutions (ab), (ac), (ad), $(abcd)$, $(abdc)$, $(acbd)$, $(acdb)$, $(adbc)$ or $(adcb)$. Since $\mathfrak{H}_{12}$ contains neither of these substitutions, this gives a contradiction.

The group $\mathfrak{H}_{13}$ has the graph H_{11} (Theorem 3. 10).

THEOREM D. 9. *The groups* $\mathfrak{H}_{16}$, $\mathfrak{H}_{20}$, $\mathfrak{H}_{21}$, $\mathfrak{H}_{22}$, $\mathfrak{H}_{23}$, $\mathfrak{H}_{24}$, $\mathfrak{H}_{29}$ *and* $\mathfrak{H}_{30}$ *have no graphs.*

The proofs, based on Theorem 4, are similar to that of Theorem C. 1.

The group $\mathfrak{H}_{17}$ has the graph H_{17} (Theorem 3. 16).
The group $\mathfrak{H}_{19}$ has the graph H_4 (Theorem 3. 3).

THEOREM D. 10. $\mathfrak{H}_{27}$ *does not have any graph.*

Proof. Suppose that $\mathfrak{H}_{27}$ has a graph H. Then H is mapped into itself by the substitution $(aebcfd)$. But then H is mapped into itself by every substitution of the group $\mathfrak{H}^* \equiv (aebcfd)_{12}$, (Theorem 4),[10] and in particular by $(ab)(cd)$. Since $\mathfrak{H}_{27}$ does not contain the latter, this gives a contradiction.

THEOREM D. 11. *The group* $\mathfrak{H}_{28}$ *has the graph*

$$P \equiv [ab, ac, ag, bc, bg, cg, de, df, dg, dh, ef, eg, eh, fg, fh].$$

The group $\mathfrak{H}_{31}$ has the graph H_2 (Theorem 3. 1).

THEOREM D. 12. $\mathfrak{H}_{32}$ *does not have any graph.* has the H_5 (Thm. 3. 4)

~~The proof is similar to that of Theorem D. 11, involving the group $\mathfrak{H}^{**} \equiv (abecfd)_{12}$ and the particular substitution $(aef)(bcd)$.[10]~~

THEOREM D. 13. $\mathfrak{H}_{33}$ *does not have any graph.*

The proof, based on Theorem 4, is similar to that of Theorem C. 1.

The group $\mathfrak{H}_{34}$ has the graph H_{18} (Theorem 3. 17).
The group $\mathfrak{H}_{36}$ does not have any graph (Theorem 5).
The group $\mathfrak{H}_{37}$ has for its graph the complete 6-point (Theorem 3).

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[10] Miller (*loc. cit.*) gives the substitutions of the group $(abcdef)_{12}$. The substitutions of $\mathfrak{H}^*$, $\mathfrak{H}^{**}$ can be obtained from these by making the obvious indicated changes in notation.